Exponential Functions

In this chapter, a will always be a positive number.

For any positive number $a > 0$, there is a function $f : \mathbb{R} \to (0, \infty)$ called an *exponential function* that is defined as $f(x) = a^x$.

For example, $f(x) = 3^x$ is an exponential function, and $g(x) = (\frac{4}{17})^x$ is an exponential function.

There is a big difference between an exponential function and a polynomial. The function $p(x) = x^3$ is a polynomial. Here the "variable", x, is being raised to some constant power. The function $f(x) = 3^x$ is an exponential function; the variable is the exponent.

Rules for exponential functions

Here are some algebra rules for exponential functions that will be explained in class.

If $n \in \mathbb{N}$, then a^n is the product of n a's. For example, $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$

$$a^0 = 1$$

If $n, m \in \mathbb{N}$, then

$$a^{\frac{n}{m}} = \sqrt[m]{a^n} = (\sqrt[m]{a})^n$$

$$a^{-x} = \frac{1}{a^x}$$

The rules above were designed so that the following most important rule of exponential functions holds:

$$a^x a^y = a^{x+y}$$

Another variant of the important rule above is

$$\frac{a^x}{a^y} = a^{x-y}$$

And there is also the following slightly related rule

$$(a^x)^y = a^{xy}$$

Examples.

- $4^{\frac{1}{2}} = \sqrt[2]{4} = 2$
- $7^{-2} \cdot 7^6 \cdot 7^{-4} = 7^{-2+6-4} = 7^0 = 1$
- $10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$
- $\frac{15^6}{15^5} = 15^{6-5} = 15^1 = 15$
- $(2^5)^2 = 2^{10} = 1024$
- $(3^{20})^{\frac{1}{10}} = 3^2 = 9$
- $8^{-\frac{2}{3}} = \frac{1}{(8)^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

The base of an exponential function

If $f(x) = a^x$, then we call a the *base* of the exponential function. The base must always be positive.

Base 1

If $f(x)$ is an exponential function whose base equals 1 – that is if $f(x) = 1^x$ – then for $n, m \in \mathbb{N}$ we have

$$f\left(\frac{n}{m}\right) = 1^{\frac{n}{m}} = \sqrt[m]{1^n} = \sqrt[m]{1} = 1$$

In fact, for any real number x, $1^x = 1$, so $f(x) = 1^x$ is the same function as the constant function $f(x) = 1$. For this reason, we usually don't talk much about the exponential function whose base equals 1.

* * * * * * * * * * * * *

Graphs of exponential functions

It's really important that you know the general shape of the graph of an exponential function. There are two options: either the base is greater than 1, or the base is less than 1 (but still positive).

Base greater than 1. If a is greater than 1, then the graph of $f(x) = a^x$ grows taller as it moves to the right. To see this, let $n \in \mathbb{Z}$. We know that $1 < a$, and we know from our rules of inequalities that we can multiply both sides of this inequality by a positive number. The positive number we'll multiply by is a^n, so that we'll have

$$a^n(1) < a^n a$$

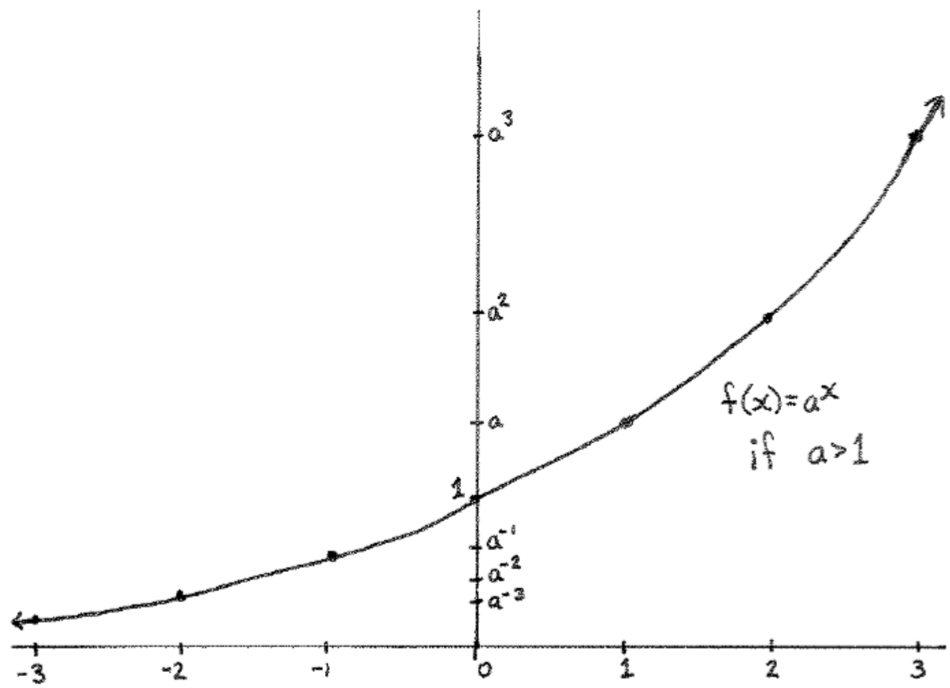
Because $a^n(1) = a^n$ and $a^n a = a^{n+1}$, the inequality above is the same as

$$a^n < a^{n+1}$$

Because the last inequality we found is true for any $n \in \mathbb{Z}$, we actually have an entire string of inequalities:

$$\cdots < a^{-3} < a^{-2} < a^{-1} < a^0 < a^1 < a^2 < a^3 < \cdots$$

Keeping in mind that a^x is positive for any number x, and that $a^0 = 1$, we now have a pretty good idea of what the graph of $f(x) = a^x$ looks like if $a > 1$. The y-intercept is at 1; when moving to the right, the graph grows taller and taller; and when moving to the left, the graph becomes shorter and shorter, shrinking towards, but never touching, the x-axis.



Not only does the graph grow bigger as it moves to the right, but it gets big in a hurry. For example, if we look at the exponential function whose base is 2, then

$$f(64) = 2^{64} = 18,446,744,073,709,525,000$$

And 2 isn't even a very big number to be using for a base (any positive number can be a base, and plenty of numbers are much, much bigger than 2). The bigger the base of an exponential function, the faster its graph grows as it moves to the right.

Moving to the left, the graph of $f(x) = a^x$ grows small very quickly if $a > 1$. Again if we look at the exponential function whose base is 2, then

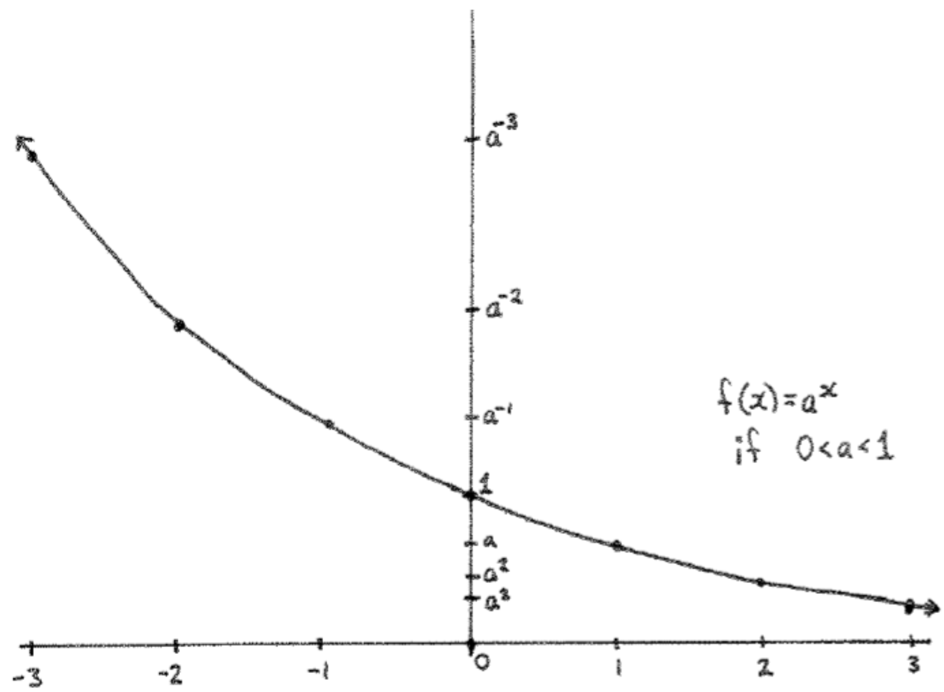
$$f(-10) = 2^{-10} = \frac{1}{2^{10}} = \frac{1}{1024}$$

The bigger the base, the faster the graph of an exponential function shrinks as it moves to the left.

Base less than 1 (but still positive). If a is positive and less than 1, then we we can show from our rules of inequalities that $a^{n+1} < a^n$ for any $n \in \mathbb{Z}$. That means that

$$\cdots > a^{-3} > a^{-2} > a^{-1} > a^0 > a^1 > a^2 > a^3 > \cdots$$

So the graph of $f(x) = a^x$ when the base is smaller than 1 slopes down as it moves to the right, but it is always positive. As it moves to the left, the graph grows tall very quickly.



One-to-one and onto

Recall that an exponential function $f : \mathbb{R} \to (0, \infty)$ has as its domain the set $\mathbb{R}$ and has as its target the set $(0, \infty)$.

We see from the graph of $f(x) = a^x$, if either $a > 1$ or $0 < a < 1$, that $f(x)$ is one-to-one and onto. Remember that to check if $f(x)$ is one-to-one, we can use the horizontal line test (which $f(x)$ passes). To check what the range of $f(x)$ is, we think of compressing the graph of $f(x)$ onto the y-axis. If we did that, we would see that the range of $f(x)$ is the set of positive numbers, $(0, \infty)$. Since the range and target of $f(x)$ are the same set, $f(x)$ is onto.

Where exponential functions appear

Exponential functions are closely related to geometric sequences. They appear whenever you are multiplying by the same number over and over and over again.

The most common example is in population growth. If a population of a group increases by say 5% every year, then every year the total population is multiplied by 105%. That is, after one year the population is 1.05 times what it originally was. After the second year, the population will be $(1.05)^2$ times what it originally was. After 100 years, the population will be $(1.05)^{100}$ times what it originally was. After x years, the population will be $(1.05)^x$ times what it originally was.

Interest rates on credit cards measure a population growth of sorts. If your credit card charges you 20% interest every year, then after 5 years of not making payments, you will owe $(1.20)^5 = 2.48832$ times what you originally charged on your credit card. After x years of not making payments, you will owe $(1.20)^x$ times what you originally charged.

Sometimes a quantity decreases exponentially over time. This process is called *exponential decay.*

If a tree dies to become wood, the amount of carbon in it decreases by 0.0121% every year. Scientists measure how much carbon is in something that died, and use the exponential function $f(x) = (0.999879)^x$ to figure out when it must have died. (The number 0.999879 is the base of this exponential function because $0.999879 = 1 - 0.000121$.) This technique is called carbon dating and it can tell us about history. For example, if scientists discover that the wood used to build a fort came from trees that died 600 years ago, then the fort was probably built 600 years ago.

* * * * * * * * * * * * *

e

Some numbers are so important in math that they get their own name. One such number is e. It is a real number, but it is not a rational number. It's very near to – but not equal to – the rational number $\frac{27}{10} = 2.7$. The importance of the number e becomes more apparent after studying calculus, but we can say something about it here.

Let's say you just bought a new car. You're driving it off the lot, and the odometer says that it's been driven exactly 1 mile. You are pulling out of the lot slowly at 1 mile per hour, and for fun you decide to keep the odometer and the speedometer so that they always read the same number.

After something like an hour, you've driven one mile, and the odometer says 2, so you accelerate to 2 miles per hour. After driving for something like a half hour, the odometer says 3, so you speed up to 3 miles an hour. And you continue in this fashion.

After some amount of time, you've driven 100 miles, so you are moving at a speed of 100 miles per hour. The odometer will say 101 after a little while, and then you'll have to speed up. After you've driven 1000 miles (and here's where the story starts to slide away from reality) you'll have to speed up to 1000 miles per hour. Now it will be just around 3 seconds before you have to speed up to 1001 miles per hour.

You're traveling faster and faster, and as you travel faster, it makes you travel faster, which makes you travel faster still, and things get out of hand very quickly, even though you started out driving at a very reasonable speed of 1 mile per hour.

If x is the number of hours you had been driving for, and $f(x)$ was the distance the car had travelled at time x, then $f(x)$ is the exponential function with base e. In symbols, $f(x) = e^x$.

Calculus studies the relationship between a function and the slope of the graph of the function. In the previous example, the function was distance travelled, and the slope of the distance travelled is the speed the car is moving at. The exponential function $f(x) = e^x$ has at every number x the same "slope" as the value of $f(x)$. That makes it a very important function for calculus.

For example, at $x = 0$, the slope of $f(x) = e^x$ is $f(0) = e^0 = 1$. That means when you first drove off the lot ($x = 0$) the odometer read 1 mile, and your speed was 1 mile per hour. After 10 hours of driving, the car will have travelled e^{10} miles, and you will be moving at a speed of e^{10} miles per hour. (By the way, e^{10} is about $22,003$.)

Exercises

For 1-11, write each number in simplest form without using a calculator, as was done in the “Examples” in this chapter. (On exams you will be asked to simplify problems like these without a calculator.)

1.) 8^{-1}

2.) $(\frac{17}{43})^3(\frac{17}{43})^{-3}$

3.) $125^{-\frac{1}{3}}$

4.) $100^{-5} \cdot 100^{457} \cdot 100^{-50} \cdot 100^{-400}$

5.) $4^{-\frac{3}{2}}$

6.) $(3^{200})^{\frac{1}{100}}$

7.) $1000^{\frac{2}{3}}$

8.) $97^{-16}\, 97^{15}$

9.) $36^{\frac{3}{6}}$

10.) $\frac{3^{297}}{3^{300}}$

11.) $(5^{\frac{4}{7}})^{\frac{14}{4}}$

For 12-20, decide which is the only number x that satisfies the given equation.

12.) $4^x = 16$

13.) $2^x = 8$

14.) $10^x = 10,000$

15.) $3^x = 9$

16.) $5^x = 125$

17.) $(\frac{1}{2})^x = 16$

18.) $(\frac{1}{4})^x = 64$

19.) $8^x = \frac{1}{4}$

20.) $27^x = \frac{1}{9}$

21.) Suppose you accidentally open a canister of plutonium in your living room and 160 units of radiation leaks out. If every year, there is half as much radiation as there was the year before, will your living room ever be free of radiation? How many units of radiation will there be after 4 years?

22.) Your uncle has an investment scheme. He guarantees that if you invest in the stock of his company, then you'll earn 10% on your money every year. If you invest \$100, and you uncle is right, how much money will you have after 20 years? (Note that when you earn 10%, you'll have 110% of what you had before.)

For 23-31, match the numbered functions with their lettered graphs.

23.) 2^x

24.) $(\frac{1}{3})^x$

25.) $2^x + 1$

26.) $(\frac{1}{3})^x - 1$

27.) $(\frac{1}{3})^{(x-1)}$

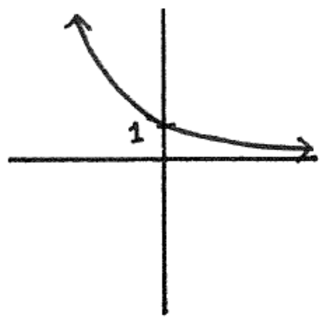
28.) -2^x

29.) $5(2^x)$

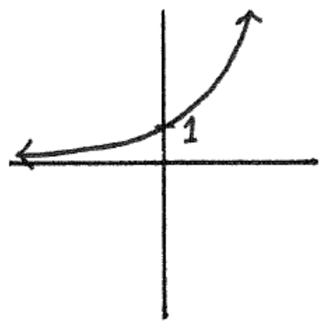
30.) $-(\frac{1}{3})^x$

31.) $2^{(x+1)}$

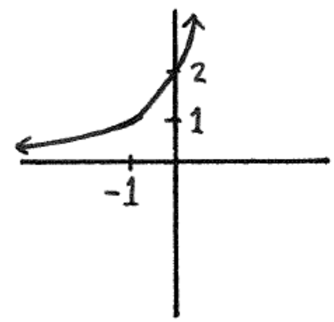
A.)



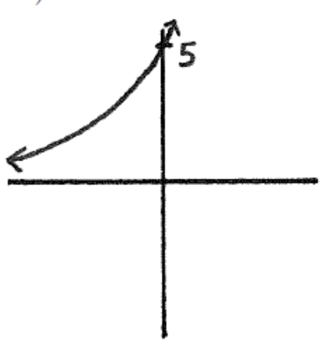
B.)



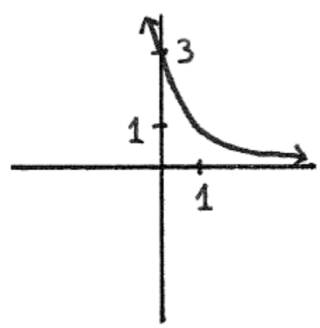
C.)



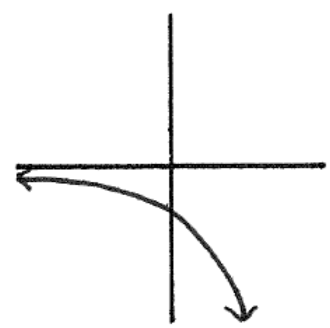
D.)



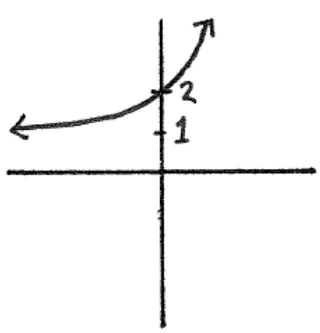
E.)



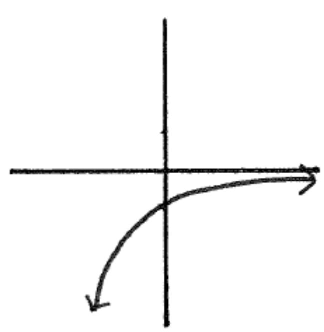
F.)



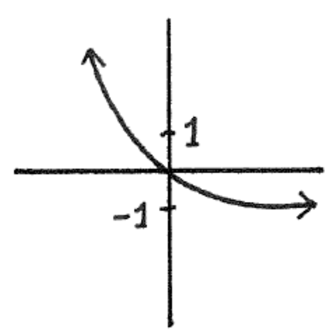
G.)



H.)



I.)



For 32-40, match the numbered functions with their lettered graphs.

32.) e^x

33.) $(\frac{1}{2})^x$

34.) $e^x - 1$

35.) $(\frac{1}{2})^x + 1$

36.) $(\frac{1}{2})^{(x+1)}$

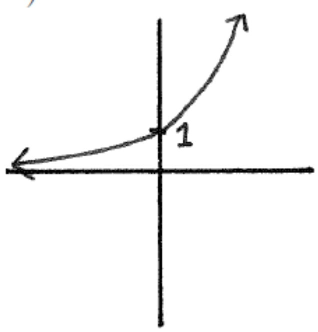
37.) $-e^x$

38.) $2(e^x)$

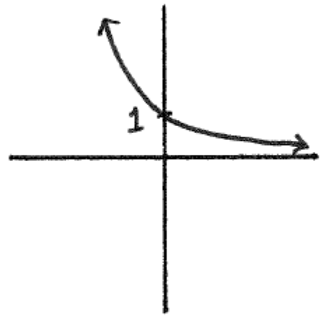
39.) $-(\frac{1}{2})^x$

40.) $e^{(x-1)}$

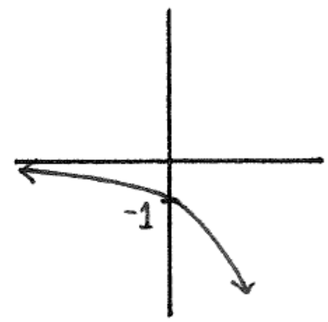
A.)



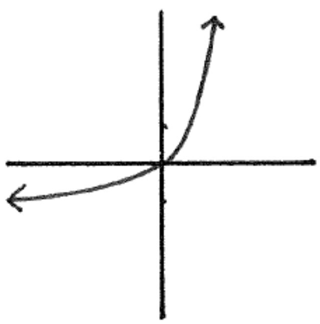
B.)



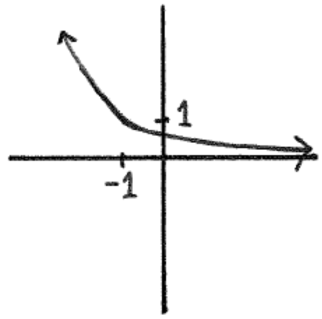
C.)



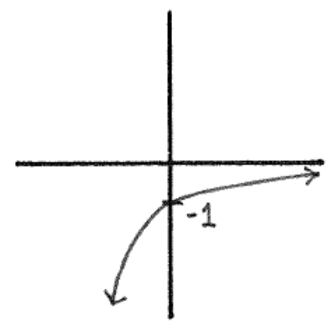
D.)



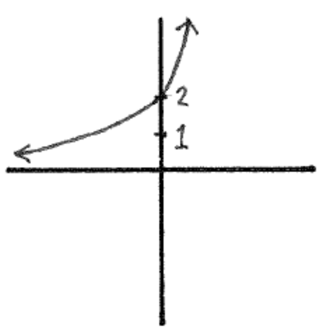
E.)



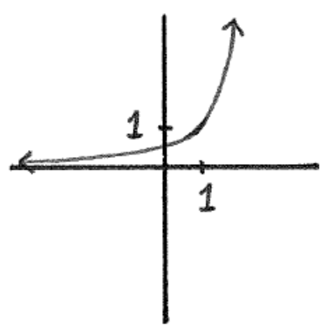
F.)



G.)



H.)



I.)