AP Calculus BC
Worksheet – Convergence Tests

Calculator OK on problems marked with an asterisk (*).

1. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ II. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ III. $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

Show which test you used to prove convergence or divergence for each series.

I $\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n} = \lim_{n\to\infty}\frac{3}{n+1} = 0$ $\boxed{\text{Converges}}$ Ratio Test

II $\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = 0$ $\boxed{\text{Converges}}$ Ratio Test

III Compare to $\sum_{n=1}^{\infty}\frac{1}{n^3}$ which converges because it is a p-series $(p>1)$.
Since $\frac{1}{n^3+1} < \frac{1}{n^3}$ on $[1,\infty)$, it must also $\boxed{\text{converge}}$.

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2. $\int_1^{\infty} \frac{x^2}{(1+x^3)^2}\,dx$ is

$u = 1+x^3$
$du = 3x^2\,dx$
$u(1)=2 \quad u(\infty)=\infty$

$\frac{1}{3}\int_2^{\infty}\frac{1}{u^2}\,du = \frac{1}{3}\left(-\frac{1}{u}\right)\Big|_2^{\infty} = \lim_{b\to\infty} -\frac{1}{3u}\Big|_2^{b}$

(A) –1/6 (B) –1/24 (C) 1/24 (D) 1/6 (E) divergent

$= \lim_{b\to\infty} -\frac{1}{3b} + \frac{1}{3(2)} = \frac{1}{6}$

3. The sum of the infinite geometric series $\frac{8}{25} - \frac{24}{125} + \frac{72}{625} - \frac{216}{3125} + \ldots$ is

(A) 0.2 (B) 0.6 (C) 0.8 (D) 1.0 (E) 1.2

$a = \frac{8}{25}$
$r = -\frac{3}{5}$

$\frac{\frac{8}{25}}{1-(-\frac{3}{5})} = \frac{\frac{8}{25}}{\frac{8}{5}} = \frac{8}{25}\cdot\frac{5}{8} = \frac{1}{5}$

$-1 < \frac{2x-1}{4} < 1$

$-4 < 2x-1 < 4$

$-3 < 2x < 5$

$-\frac{3}{2} < x < \frac{5}{2}$

4. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n \cdot 4^n}$ converges?

(A) $-\frac{3}{2} \le x \le \frac{5}{2}$ (B) $-\frac{3}{2} \le x < \frac{5}{2}$ (C) $-\frac{3}{2} < x \le \frac{5}{2}$

(D) $-3 \le x < 5$ (E) $-3 < x \le 5$

$\lim_{n\to\infty} \frac{|2x-1|^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n \cdot 4^n}{|2x-1|^n} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{|2x-1|}{4} = \frac{|2x-1|}{4}$

$x = \frac{5}{2} \to \sum_{n=1}^{\infty} \frac{4^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges $(p=1)$

$x = -\frac{3}{2} \to \sum_{n=1}^{\infty} \frac{(-4)^n}{n(4)^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Converges (alternating series)

5. Which of the following series converge?

(II) $\sum_{n=1}^{\infty} \frac{\sin\left[(2n-1)\frac{\pi}{2}\right]}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ Alternating Series, so converges

I. $\sum_{n=1}^{\infty} \frac{n}{n+3}$ II. $\sum_{n=1}^{\infty} \frac{\sin\left[(2n-1)\frac{\pi}{2}\right]}{n}$ III. $\sum_{n=1}^{\infty} \frac{2}{n}$

Diverges $\left(\lim_{n\to\infty} \frac{n}{n+3} = 1\right)$ Converges Diverges (p-series, $p=1$)

(A) None (B) II only (C) III only (D) I and II (E) I and III

6. If $\sum_{n=0}^{\infty} a_n x^n$ is a Taylor series that converges to $f(x)$ for all real x, then $f'(x)$ is

(A) 0 (B) a_1 (C) $\sum_{n=0}^{\infty} n a_n x^{n-1}$

(D) $\sum_{n=0}^{\infty} n(a_n)^{n-1}$ (E) nx^{n-1}

*7. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n}$ converges?

(A) $-2 < x < 4$ (B) $-2 \le x < 4$ (C) $-2 \le x < 4$

(A) $-1 < x < 1$ (A) $-1 \le x \le 1$

Ratio Test:

$\lim_{n\to\infty} \frac{|x-1|^{n+1}}{3^{n+1}} \cdot \frac{3^n}{|x-1|^n} = \lim_{n\to\infty} \frac{|x-1|}{3} = \frac{|x-1|}{3}$

$-1 < \frac{x-1}{3} < 1$ $-2 < x < 4$

$-3 < x-1 < 3$

$x = -2: \sum_{n=1}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n$ diverges

$x = 4: \sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1^n$ diverges

*8. For what values of k will $\sum_{n=1}^{\infty}\left(\frac{k}{3}\right)^n$ converge?

$\lim_{n\to\infty} \frac{\left(\frac{k}{3}\right)^{n+1}}{\left(\frac{k}{3}\right)^n} = \lim_{n\to\infty} \frac{k}{3} = \frac{k}{3}$

$-1 < \frac{k}{3} < 1$

$-3 < k < 3$

(A) $k > 3$ (B) $k < -3$ (C) $k < 3$

(D) $-3 \le k \le 3$ (E) $-3 < k < 3$

$k=3 \to \sum_{n=1}^{\infty}\left(\frac{3}{3}\right)^n \to$ diverges $\quad k=-3 \to \sum_{n=1}^{\infty}\left(\frac{-3}{3}\right)^n = \sum_{n=1}^{\infty}(-1)^n \to$ diverges

*9. Let f be the function given by $f(x) = x^2 - 3x + 5$. The tangent line to the graph of f at $x = 3$ is used to approximate values of $f(x)$. Which of the following is the greatest value of x for which the absolute value of the error resulting from this tangent line approximation is less than 0.5?

(A) 3.4 (B) 3.5 (C) 3.6 (D) 3.7 (E) 3.8

$f(3)=5$

$f'(x)=2x-3$

$f'(3)=3$

$y-5=3(x-3)$

$y=3x-4$

$(x^2-3x+5)-(3x-4)<0.5$

Use calculator $\to$ error $=0.5$ at $x \approx 3.707$

10. Find the interval of convergence and radius of convergence of $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^n(n+1)}$.

Ratio Test:

$\lim_{n\to\infty} \frac{|x^{n+1}|}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{|x^n|} = \lim_{n\to\infty} \frac{|x|}{3} \cdot \frac{n+1}{n+2} = \frac{|x|}{3}$ $\quad \frac{|x|}{3} < 1 \quad |x| < \frac{1}{3}$ $\boxed{R=\frac{1}{3}}$

$-\frac{1}{3} < x < \frac{1}{3}$

$x=\frac{1}{3} \to \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{3}\right)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{9^n(n+1)}$ Converges (alternating series)

$x=-\frac{1}{3} \to \sum_{n=0}^{\infty}\frac{(-1)^n\left(-\frac{1}{3}\right)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}\frac{1}{9^n(n+1)}$ Compare to $\frac{1}{9^n}$, which converges (ratio test). Since $\frac{1}{9^n(n+1)} \le \frac{1}{9^n}$ on $[0,\infty)$, also converges

$\boxed{-\frac{1}{3} \le x \le \frac{1}{3}}$

11. Prove whether $\sum_{k=0}^{\infty}\frac{2k^2+1}{2k^{8/3}-1}$ converges or diverges using a p-series test.

Compare to $\sum_{n=0}^{\infty}\frac{1}{n^{2/3}}$ which is a divergent p-series ($p \le 1$)

Since $\sum_{k=0}^{\infty}\frac{2k^2+1}{2k^{8/3}-1} \ge \sum_{n=0}^{\infty}\frac{1}{n^{2/3}}$ on $[0,\infty)$ it also diverges.

12. Use the alternating series test to show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$ converges.

(1) $\frac{n+3}{n(n+1)} > 0$ ✓

(2) $a_n > a_{n+1}$

$\frac{n+3}{n(n+1)} > \frac{n+4}{(n+1)(n+2)}$ ✓

(3) $\lim_{n\to\infty}\frac{n+3}{n(n+1)} \to 0$ ✓

All 3 conditions of alternating series test are satisfied, so series converges.

13. The function g has derivatives of all orders, and the Maclaurin series for g is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \ldots$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

$$\lim_{n\to\infty} \frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}} = \lim_{n\to\infty} x^2 \cdot \frac{2n+3}{2n+5} = x^2$$

$$-1 < x^2 < 1 \qquad -1 < x < 1$$

$x=1 \Rightarrow \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+3} \rightarrow$ converges because of alt. series test

$x=-1 \Rightarrow \sum_{n=0}^{\infty}(-1)^n \frac{(-1)^{2n+1}}{2n+3} \rightarrow$ converges because of alt. series test

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

Use 3rd term: $\frac{\left(\frac{1}{2}\right)^5}{7} < \frac{1}{200} \qquad \frac{1}{224} < \frac{1}{200}$

Error must be less than the third term. Since third term is $\frac{1}{224}$ which is less than $\frac{1}{200}$, error must be less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

$$\frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7} + \ldots + \frac{(-1)^n(2n+1)x^{2n}}{2n+3} + \ldots$$