AP CALC. BC   Section 10.3: **PARAMETRIC EQUATIONS AND CALCULUS**, pg. 719 – **DAY 1 AND 2**

TANGENTS WITH PARAMETRIC EQUATIONS

THEOREM 10.7 Parametric Form of the Derivative

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

Sample Problem #1:

Find $\frac{dy}{dx}$ for the curve given by $x = \sin(t)$ and $y = \cos(t)$.

$$\frac{dy}{dx} = \frac{-\sin(t)}{\cos(t)} = \boxed{-\tan(t)} = \boxed{-\frac{x}{y}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\sec^2(t)}{\cos(t)} = \boxed{-\frac{1}{\cos^3(t)}} = \boxed{-\frac{1}{y^3}}$$

Because $\frac{dy}{dt}$ is a function of t, you can use the rule above repeatedly to find *higher-order derivatives.* For instance:

$$\boxed{\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} \\ \frac{d^3y}{dx^3} &= \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{\frac{dx}{dt}} \end{aligned}}$$

LET'S DETERMINE $\frac{d^2y}{dx^2}$ FOR THE PARAMETRIC EQUATION ABOVE.

Sample Problem #2:

For the curve given by $x=\sqrt{t}$ and $y=\frac{1}{4}\left(t^2-4\right)$, $t \geq 0$ find the slope and concavity at the point $(2,3)$.

$$\frac{dy}{dx} = \frac{\frac{1}{4}(2t)}{\frac{1}{2\sqrt{t}}} = \frac{t/\cancel{2}}{\frac{1}{\cancel{2}\sqrt{t}}} = \frac{t}{\frac{1}{\sqrt{t}}} = t\sqrt{t}$$

$x=2 \Rightarrow 2=\sqrt{t} \Rightarrow t=4$

$y=3 \Rightarrow 3=\frac{1}{4}(t^2-4) \Rightarrow t^2-4=12 \Rightarrow t^2=16 \Rightarrow t=4$

$$m=\frac{dy}{dx}(2,3)=\left.\frac{dy}{dx}\right|_{t=4} = 4\sqrt{4} = 8 \qquad \boxed{m=8}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{t}+t/2\sqrt{t}}{1/2\sqrt{t}} = \frac{\frac{3t}{2\sqrt{t}}}{\frac{1}{2\sqrt{t}}} = 3t$$

at $t=4 \Rightarrow \frac{d^2y}{dx^2} > 0 \Rightarrow$ Concave Up.

Sample Problem #3:

Find the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \quad \text{and} \quad y = t^2 \quad \text{at } (0,4)$$

$x=0 \Rightarrow t^5 - 4t^3 = t^3(t^2-4) = 0 \Rightarrow t = 0, \pm 2$

$y=4 \Rightarrow t^2 = 4 \Rightarrow t = \pm 2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2}$$

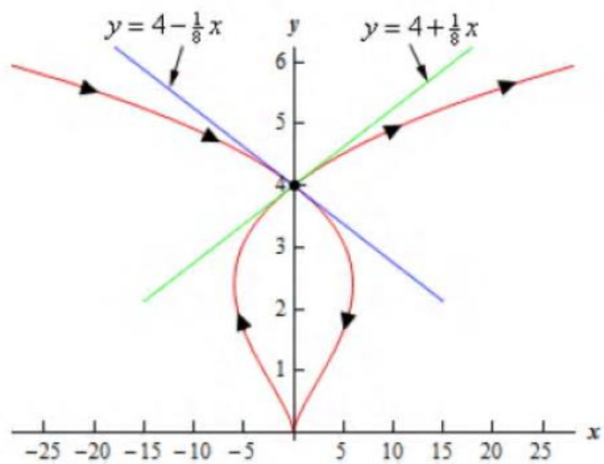
$t=-2: m = \frac{-4}{32} = -1/8$

$t=2: m = \frac{4}{32} = 1/8$

Tangent Lines to the curve at (0,4)

$y = -1/8\,x + 4$ and $y = 1/8\,x - 4$

A quick graph of the parametric curve will explain what is going on here.



So, the parametric curve crosses itself! That explains how there can be more than one tangent line. There is one tangent line for each instance that the curve goes through the point.

Horizontal tangents will occur where the derivative is zero and that means that we'll get horizontal tangent at values of *t* for which we have $\frac{dy}{dt} = 0$.

Horizontal Tangent for Parametric Equations

$$\frac{dy}{dt} = 0, \quad \text{provided} \quad \frac{dx}{dt} \neq 0$$

Vertical tangents will occur where the derivative is not defined and so we'll get vertical tangents at values *t* for which we have $\frac{dx}{dt} = 0$

Vertical Tangent for Parametric Equations

$$\frac{dx}{dt} = 0, \quad \text{provided} \quad \frac{dy}{dt} \neq 0$$

Sample Problem #4:

Determine the *x*-*y* coordinates of the points where the following parametric equations will have **horizontal or vertical tangents**.

$$x(t) = t^3 - 3t \qquad \text{and} \qquad y = 3t^2 - 9$$

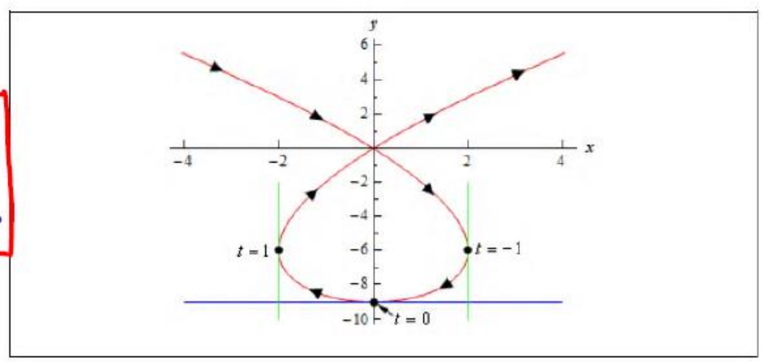
$\frac{dy}{dt} = 12t = 0 \Rightarrow t = 0 \Rightarrow$ Horizontal Tangent Line at $(0,-9)$

$\frac{dx}{dt} = 3t^2 - 3 = 0 \Rightarrow t^2 - 1 = 0 \Rightarrow t = \pm 1$

$t=1$: $x = -2$, $y = -6$

$t=-1$: $x = 2$, $y = -6$

Vertical Tangent Lines at: $(-2,-6)$ and $(2,-6)$.



Sample Problem #5:

Determine the values of t for which the parametric curve given by the following set of parametric equations is concave up and concave down.

$$x = 1 - t^2 \qquad \text{and} \qquad y = t^7 + t^5$$

$$\frac{dy}{dx} = \frac{7t^6 + 5t^4}{-2t} = -\frac{7}{2}t^5 - \frac{5}{2}t^3$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{35}{2}t^4 - \frac{15}{2}t^2}{-2t} = \frac{35}{4}t^3 + \frac{15}{4}t = 0$$

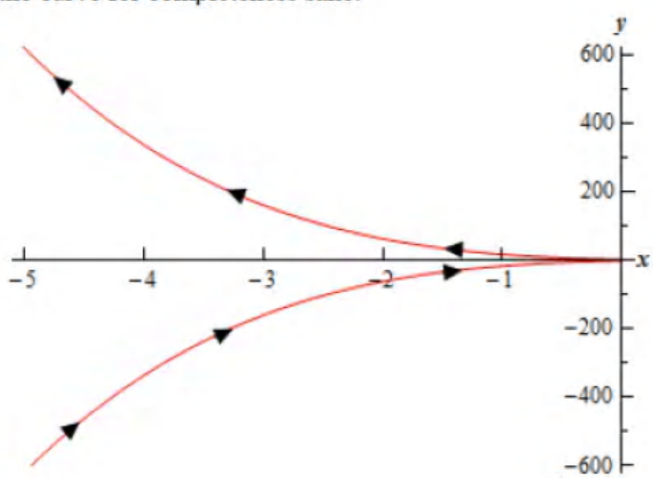
$$35t^3 + 15t = 0$$

$$5t(7t^2 + 3) = 0$$

$$t = 0$$

Concave Down for: $t<0$ and Concave Up for: $t>0$

Here is a sketch of the curve for completeness sake.



AREA WITH PARAMETRIC EQUATIONS

In this section we will find a formula for determining the area under a parametric curve given by the parametric equations,

$$x = f(t) \qquad\qquad y = g(t)$$

We will also need to further add in the assumption that the curve is traced out exactly once as t increases from α to β.

We will do this in much the same way that we found the first derivative in the previous section. We will first recall how to find the area under $y = F(x)$ on $a \le x \le b$.

$$A = \int_a^b F(x)\,dx$$

We will now think of the parametric equation $x = f(t)$ as a substitution in the integral. We will also assume that $a = f(\alpha)$ and $b = f(\beta)$ for the purposes of this formula. There is actually no reason to assume that this will always be the case and so we'll give a corresponding formula later if it's the opposite case ($b = f(\alpha)$ and $a = f(\beta)$).

So, if this is going to be a substitution we'll need,

$$dx = f'(t)\,dt$$

Plugging this into the area formula above and making sure to change the limits to their corresponding t values gives us,

$$A = \int_\alpha^\beta F(f(t))\,f'(t)\,dt$$

Since we don't know what $F(x)$ is we'll use the fact that

$$y = F(x) = F(f(t)) = g(t)$$

and we arrive at the formula that we want.

Area Under Parametric Curve, Formula I

$$A = \int_\alpha^\beta g(t)\,f'(t)\,dt$$

Now, if we should happen to have $b = f(\alpha)$ and $a = f(\beta)$ the formula would be,

Area Under Parametric Curve, Formula II

$$A = \int_\beta^\alpha g(t)\,f'(t)\,dt$$

Sample Problem #6:

Determine the area under the parametric curve given by the following parametric equations.

$$x = 6(\theta - \sin\theta) \qquad y = 6(1-\cos\theta) \qquad 0 \le \theta \le 2\pi$$

$$A = \int_{\alpha}^{\beta} g(t)\, f'(t)\, dt$$

$$A = \int_0^{2\pi} 6(1-\cos\theta)\cdot 6(1-\cos\theta)\, d\theta$$

$$A = 36\int_0^{2\pi} (1-\cos\theta)^2\, d\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$A = 36\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\, d\theta$$

$$A = 36\int_0^{2\pi} 1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta)\, d\theta$$

$$A = 36\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi}$$

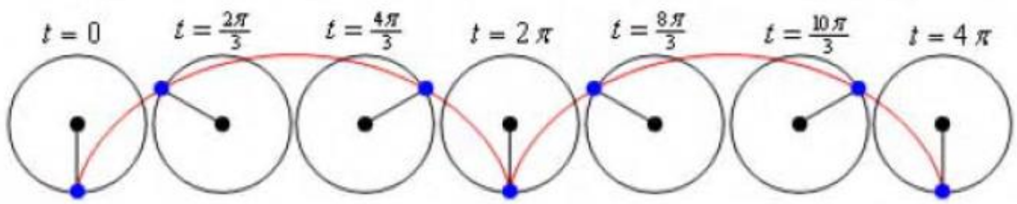
$$A = 36\left[3\pi\right] = \boxed{108\pi}$$

The parametric curve (without the limits) we used in the previous example is called a **cycloid**. In its general form the cycloid is,

$$x = r(\theta - \sin\theta) \qquad\qquad y = r(1 - \cos\theta)$$

The cycloid represents the following situation. Consider a wheel of radius r. Let the point where the wheel touches the ground initially be called P. Then start rolling the wheel to the right. As the wheel rolls to the right trace out the path of the point P. The path that the point P traces out is called a cycloid and is given by the equations above. In these equations we can think of θ as the angle through which the point P has rotated.

Here is a cycloid sketched out with the wheel shown at various places. The blue dot is the point P on the wheel that we're using to trace out the curve.



From this sketch we can see that one arch of the cycloid is traced out in the range $0 \le \theta \le 2\pi$. This makes sense when you consider that the point P will be back on the ground after it has rotated through and angle of 2π.

ARC LENGTH WITH PARAMETRIC EQUATIONS

In this section we will look at the arc length of the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

We will also be assuming that the curve is traced out exactly once as t increases from α to β. We will also need to assume that the curve is traced out from left to right as t increases. This is equivalent to saying,

$$\frac{dx}{dt} \ge 0 \qquad \text{for } \alpha \le t \le \beta$$

So, let's start out the derivation by recalling the arc length formula as we first derived it in the arc length section of the Applications of Integrals chapter.

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

We will use the first ds above because we have a nice formula for the derivative in terms of the parametric equations (see the Tangents with Parametric Equations section). To use this we'll also need to know that,

$$dx = f'(t)\,dt = \frac{dx}{dt}dt$$

The arc length formula then becomes,

$$L = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\ \frac{dx}{dt}dt = \int_\alpha^\beta \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\ \frac{dx}{dt}dt$$

This is a particularly unpleasant formula. However, if we factor out the denominator from the square root we arrive at,

$$L = \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; \frac{dx}{dt} dt$$

Now, making use of our assumption that the curve is being traced out from left to right we can drop the absolute value bars on the derivative which will allow us to cancel the two derivatives that are outside the square root this gives,

Arc Length for Parametric Equations

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

Notice that we could have used the second formula for ds above is we had assumed instead that

$$\frac{dy}{dt} \geq 0 \qquad \text{for } \alpha \leq t \leq \beta$$

If we had gone this route in the derivation we would have gotten the same formula.

Sample Problem #7:

Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(t) \qquad y = 3\cos(t) \qquad 0 \leq \theta \leq 2\pi$$

$$L = \int_0^{2\pi} \sqrt{9\cos^2 t + 9\sin^2 t} \, dt = \int_0^{2\pi} 3 \, dt = 3t \Big]_0^{2\pi} = \boxed{6\pi}$$

Since this is a circle we could have just used the fact that the length of the circle is just the circumference of the circle. This is a nice way, in this case, to verify our result.

LET'S TAKE A LOOK AT ONE POSSIBLE CONSEQUENCE OF A CURVE BEING TRACED OUT MORE THAN ONCE, AND US TRYING TO FIND THE LENGTH OF THE CURVE WITHOUT TAKING THIS INTO CONSIDERATION.

Sample Problem #8:

Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(3t) \qquad y = 3\cos(3t) \qquad 0 \le \theta \le 2\pi$$

$$L = \int_0^{2\pi} \sqrt{81\cos^2(3t) + 81\sin^2(3t)}\, dt = \int_0^{2\pi} 9\, dt = 9t\Big]_0^{2\pi} = \boxed{18\pi}$$

The answer we got form the arc length formula in this example was 3 times the actual length. Recalling that we also determined that this circle would trace out three times in the range given, the answer should make some sense.

If we had wanted to determine the length of the circle for this set of parametric equations we would need to determine a range of t for which this circle is traced out exactly once. This is, $0 \le t \le \frac{2\pi}{3}$. Using this range of t we get the following for the length.

$$\begin{aligned} L &= \int_0^{\frac{2\pi}{3}} \sqrt{81\sin^2(t) + 81\cos^2(t)}\, dt \\ &= \int_0^{\frac{2\pi}{3}} 9\, dt \\ &= 6\pi \end{aligned}$$

which is the correct answer.

Be careful to not make the assumption that this is always what will happen if the curve is traced out more than once. Just because the curve traces out n times does not mean that the arc length formula will give us n times the actual length of the curve!

SURFACE AREA WITH PARAMETRIC EQUATIONS

In this final section of looking with calculus application with parametric equations we will take a look at determining the surface area of a region obtained by rotating a parametric curve about the x or y-axis.

We will rotate the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

about the x or y-axis. We are going to assume that the curve is traced out exactly once as t increases from α to β. At this point there actually isn't all that much to do. We know that the surface area can be found by using one of the following two formulas depending on the axis of rotation (recall the Surface Area section of the Applications of Integrals chapter).

$$S = \int 2\pi y \, ds \qquad \text{rotation about } x-\text{axis}$$

$$S = \int 2\pi x \, ds \qquad \text{rotation about } y-\text{axis}$$

All that we need is a formula for ds to use and from the previous section we have,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \qquad \text{if } x = f(t), y = g(t), \alpha \le t \le \beta$$

which is exactly what we need.

We will need to be careful with the x or y that is in the original surface area formula. Back when we first looked at surface area we saw that sometimes we had to substitute for the variable in the integral and at other times we didn't. This was dependent upon the ds that we used. In this case however, we will always have to substitute for the variable. The ds that we use for parametric equations introduces a dt into the integral and that means that everything needs to be in terms of t. Therefore, we will need to substitute the appropriate parametric equation for x or y depending on the axis of rotation.

Sample Problem #9:

Determine the surface area of the solid obtained by rotating the following parametric curve about the x-axis.

$$x = \cos^3(\theta) \qquad y = \sin^3(\theta) \qquad 0 \le \theta \le \frac{\pi}{2}$$

$\frac{dx}{dt} = -3\cos^2\theta\sin\theta$

$S = \int 2\pi y \, ds$ rotation about x – axis

$S = \int 2\pi x \, ds$ rotation about y – axis

$\frac{dy}{dt} = 3\sin^2\theta\cos\theta$

All that we need is a formula for ds to use and from the previous section we have,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{if } x = f(t),\ y = g(t),\ \alpha \le t \le \beta$$

$\frac{ds}{dt} = \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}$

$\frac{ds}{dt} = \sqrt{9\cos^2\theta\sin^2\theta\,(\cos^2\theta + \sin^2\theta)}$

$\frac{ds}{dt} = 3\,|\cos\theta\sin\theta| = 3\cos\theta\sin\theta$ (the parameter is on the I quadrant)

$$S = \int_0^{\pi/2} 2\pi\sin^3\theta \cdot 3\cos\theta\sin\theta\,d\theta = 6\pi\int_0^{\pi/2} \sin^4\theta\cos\theta\,d\theta$$

$$S = 6\pi\int_0^1 u^4\,du = 6\pi \cdot \frac{u^5}{5}\Big]_0^1 = \boxed{\frac{6\pi}{5}}$$