MATH 1020 WORKSHEET 10.2
Calculus with Parametric Curves

This sections included formulas for slope and concavity of parametric curves and for arc length when given parametric curves.

Find dy/dx and d^2y/dx^2 for the parametric equations $x = \sqrt{t}$, $y = \sqrt{t-1}$ and evaluate each at $t = 2$.

Solution. For both the slope and concavity, one first needs to determine dx/dt and dy/dt before proceeding.

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{t-1}}$$

Next one applies the formula for the slope in parametric form

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{ft}} \\ &= \frac{\frac{1}{2\sqrt{t-1}}}{\frac{1}{2\sqrt{t}}} \\ &= \frac{\sqrt{t}}{\sqrt{t-1}} \\ &= \sqrt{\frac{t}{t-1}} \end{aligned}$$

Thus

$$\boxed{\left.\frac{dy}{dx}\right|_{t=2} = \sqrt{2}}.$$

To determine the concavity, we apply to parametric formula

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}\left(\sqrt{\frac{t}{t-1}}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{1}{2}\left(\frac{t}{t-1}\right)^{-1/2}\left(\frac{(t-1)-t}{(t-1)^2}\right)}{\frac{1}{2\sqrt{t}}} \\ &= 2\sqrt{t}\cdot\left(\frac{1}{2}\left(\frac{t}{t-1}\right)^{-1/2}\left(\frac{-1}{(t-1)^2}\right)\right) \end{aligned}$$

Evaluating at $t = 2$ one gets

$$\begin{aligned} \left.\frac{d^2y}{dx^2}\right|_{t=2} &= 2\sqrt{2}\cdot\left(\frac{1}{2}\left(\frac{2}{1}\right)^{-1/2}\left(\frac{-1}{1^2}\right)\right) \\ &= 2\sqrt{2}\cdot\left(\frac{1}{2}\frac{1}{\sqrt{2}}(-1)\right) \\ &= -1 \end{aligned}$$

Thus

$$\boxed{\left.\frac{d^2y}{dx^2}\right|_{t=2} = -1}.$$

Find the arc length of the curve given by $x = t$, $y = \frac{t^5}{10} + \frac{1}{6t^3}$ on the interval $1 \leq t \leq 2$.

Solution. One first needs to determine dx/dt and dy/dt before proceeding.

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = \frac{t^4}{2} - \frac{1}{2t^4}$$

Next one applies the formula for the arc length in parametric form

$$\begin{aligned}
L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_1^2 \sqrt{(1)^2 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\, dt \\
&= \int_1^2 \sqrt{1 + \left(\frac{t^8}{4} - 2 \cdot \frac{t^4}{2} \cdot \frac{1}{2t^4} + \frac{1}{4t^8}\right)}\, dt \\
&= \int_1^2 \sqrt{1 + \left(\frac{t^8}{4} - \frac{1}{2} + \frac{1}{4t^8}\right)}\, dt \\
&= \int_1^2 \sqrt{\left(\frac{t^8}{4} + \frac{1}{2} + \frac{1}{4t^8}\right)}\, dt \\
&= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2}\, dt \\
&= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right) dt \\
&= \left(\frac{t^5}{10} - \frac{1}{6t^3}\right)\Bigg|_1^2 \\
&= (\frac{32}{10} - \frac{1}{10}) - (\frac{1}{48} - \frac{1}{6}) \\
&= \frac{31}{10} - \frac{-7}{8} \\
&= \frac{744 + 35}{240}
\end{aligned}$$

Thus

$$\boxed{\int_1^2 \sqrt{(1)^2 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\, dt = \frac{779}{24}}.$$