AP CALC. BC Section 10.3: **PARAMETRIC EQUATIONS AND CALCULUS**, pg. 71

HOMEWORK: DAY 1 – ODDS, DAY 2 – EVENS

Answers at end

In the following exercises, find dy/dx and d^2y/dx^2 and evaluate each at the indicated value of the parameter.

1. $x = 3t$ and $y = 4t + 1$ $\quad t = 2$

2. $x = \sqrt{t}$ and $y = 4t - 1$ $\quad t = 1$

3. $x = 2t - 2$ and $y = t^2 - 4t$ $\quad t = 1$

4. $x = 2\cos t$ and $y = 2\sin t$ $\quad t = \frac{\pi}{4}$

5. $x = \sqrt{t}$ and $y = \sqrt{t-1}$ $\quad t = 2$

6. $x = \cos^3 t$ and $y = \sin^3 t$ $\quad t = \frac{\pi}{4}$

In the following 2 exercises, find the equation of the tangent line at the indicated points on the curve.

7. $x = \dfrac{2}{\tan\theta}$ and $y = 2\sin^2\theta$ at $\left(2\sqrt{3}, \frac{1}{2}\right)$ at $(2,1)$

8. $x = -4\cos\theta$ and $y = 3 + 2\sin\theta$ at $\left(-2, 3+\sqrt{3}\right)$

In the following exercises, find all points of horizontal and vertical tangency to the curve.

9. $x = 2 - t$ and $y = t^2$

10. $x = t + 4$ and $y = t^2 + 3t$

11. $x = 8\cos^2\theta$ and $y = 4\sin\theta$

12. $x = 2\theta$ and $y = 2(1 - \cos\theta)$

13. $x = t^2 + t - 2$ and $y = t^3 - 3t$

14. $x = 2\cos\theta$ and $y = 2\sin 2\theta$

Find the arc length of the given curve on the indicated interval. Calculators permitted on 15 and 16.

15. $x = t^2$ and $y = 4t^3 - 1$ $\quad [-1,1]$

16. $x = \sqrt{t}$ and $y = 3t - 1$ $\quad [1,2]$

17. $x = e^{-t}\cos t$ and $y = e^{-t}\sin t$ $\left[0, \frac{\pi}{2}\right]$

18. $x = t$ and $y = \dfrac{t^5}{10} + \dfrac{1}{6t^3}$ $[1,2]$

19. The path of a projectile is modeled by the equations $x = (100\cos 30^\circ)t$ and $y = (100\sin 30^\circ)t - 16t^2$ where x and y are measured in feet. Graph the path of the projectile and use the integration capabilities of the calculator to approximate the arc length of the path and the difference between the arc length and the range of the projectile.

20. Given the parametric equations $x = \dfrac{4t}{1+t^3}$ and $y = \dfrac{4t^2}{1+t^3}, t \geq 0$, sketch it on your calculator.

a) Determine what values of t give a horizontal tangent.

b) Approximate the arc length of the closed loop. Set up the integral and use your calculator appropriately

21. Find the area of the surface generated by revolving the curve about the given axes. Use calculator on a).

a) $x = t$ and $y = 8 - 4t$, $[0,2]$
i) x – axis ii) y – axis

b. $x = 6\cos\theta$ and $y = 6\sin\theta$, $\left[0, \frac{\pi}{2}\right]$
i) x – axis ii) y – axis

Answers

HOMEWORK- Answer Key! STUDY

1. $\frac{dy}{dx} = \boxed{\frac{4}{3}}$

$\frac{d^2y}{dx^2} = \frac{0}{3} = \boxed{0}$

2. $\frac{dy}{dx} = \frac{4}{\frac{1}{2\sqrt{t}}} = \boxed{8\sqrt{t}}$ $\quad \left.\frac{dy}{dx}\right|_{t=1} = \boxed{8}$

$\frac{d^2y}{dx^2} = \frac{\frac{4}{\sqrt{t}}}{\frac{1}{2\sqrt{t}}} = \boxed{8}$

3. $\frac{dy}{dx} = \frac{2t-4}{2} = \boxed{t-2}$ $\quad \left.\frac{dy}{dt}\right|_{t=1} = \boxed{-1}$

$\frac{d^2y}{dx^2} = \boxed{\frac{1}{2}}$

4. $\frac{dy}{dx} = \frac{2\cos t}{-2\sin t} = \boxed{-\cot t}$

$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \boxed{-1}$

$\frac{d^2y}{dx^2} = \frac{-\csc^2 t}{-2\sin t} = \boxed{\frac{1}{2\sin^3 t}}$

$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/4} = \frac{1}{2\frac{\sqrt{2}}{4}} = \boxed{\sqrt{2}}$

5. $\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{t-1}}}{\frac{1}{2\sqrt{t}}} = \boxed{\frac{\sqrt{t}}{\sqrt{t-1}}}$

$\left.\frac{dy}{dx}\right|_{t=2} = \boxed{\sqrt{2}}$

$$\frac{d^2y}{dx^2} = \frac{\frac{\frac{1}{2\sqrt{t}}\sqrt{t-1} - \sqrt{t}\frac{1}{2\sqrt{t-1}}}{t-1}}{\frac{1}{2\sqrt{t}}} = \frac{\frac{\frac{\cancel{t}-1-\cancel{t}}{2\sqrt{t(t-1)}}}{t-1}}{\frac{1}{2\sqrt{t}}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{-1}{\cancel{2\sqrt{t}}(t-1)^{3/2}}}{\frac{1}{\cancel{2\sqrt{t}}}} = \boxed{-\frac{1}{(t-1)^{3/2}}}$$

$\left.\frac{d^2y}{dx^2}\right|_{t=2} = \boxed{-1}$

6. $\dfrac{dy}{dx} = \dfrac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} = -\dfrac{\sin t}{\cos t} = \boxed{-\tan t}$ $\qquad \left.\dfrac{dy}{dx}\right|_{t=\pi/4} = \boxed{-1}$

$\dfrac{d^2y}{dx^2} = \dfrac{-\sec^2 t}{-3\cos^2 t \sin t} = \boxed{\dfrac{1}{3\cos^4 t \sin t}}$ $\qquad \left.\dfrac{d^2y}{dx^2}\right|_{t=\pi/4} = \boxed{\dfrac{8}{3\sqrt{2}}}$

7. $\dfrac{dy}{dx} = \dfrac{4\sin\theta\cos\theta}{-\dfrac{2\sec^2\theta}{\tan^2\theta}} = -2\sin\theta\cos\theta \cdot \dfrac{\sin^2\theta}{\cancel{\cos^2\theta}} \cdot \cancel{\cos^2\theta}$

$\dfrac{dy}{dx} = -2\sin^3\theta\cos\theta$

a) $(2\sqrt{3}, \frac{1}{2}) \Rightarrow x = \dfrac{2}{\tan\theta} = 2\sqrt{3} \Rightarrow \tan\theta = \dfrac{\sqrt{3}}{3} \Rightarrow \theta = \pi/6,\ \theta = 7\pi/6$

$y = 2\sin^2\theta = \frac{1}{2} \Rightarrow \sin^2\theta = \frac{1}{4} \Rightarrow \theta = \pi/6,\ 5\pi/6,\ 7\pi/6,\ 11\pi/6$

$\theta = \pi/6$ or $\theta = 7\pi/6$

$m = -\cancel{2}\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{3}}{\cancel{2}}\right)$ $\qquad m = -\cancel{2}\left(-\frac{1}{2}\right)^3\left(-\frac{\sqrt{3}}{\cancel{2}}\right)$

$m = -\dfrac{\sqrt{3}}{8}$ $\qquad m = \dfrac{\sqrt{3}}{8}$

$\boxed{y - \frac{1}{2} = -\frac{\sqrt{3}}{8}\left(x - 2\sqrt{3}\right)}$ $\qquad \boxed{y - \frac{1}{2} = \frac{\sqrt{3}}{8}\left(x - 2\sqrt{3}\right)}$

b) $(2,1) \Rightarrow \quad x = \frac{2}{\tan\theta} = 2 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \pi/4,\ 5\pi/4$

$y = 2\sin^2\theta = 1 \Rightarrow \sin^2\theta = 1/2 \Rightarrow \sin\theta = \pm\frac{\sqrt{2}}{2} \Rightarrow \theta = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$

$$\frac{dy}{dx} = -2\sin^3\theta\cos\theta$$

$\theta = \pi/4$

$m = -2\left(\frac{\sqrt{2}}{2}\right)^4 = -1/2$

$\theta = 5\pi/4$

$m = -2\left(-\frac{\sqrt{2}}{2}\right)^4 = -1/2$

$$\boxed{y - 1 = -1/2\,(x-1)}$$

8. $\frac{dy}{dx} = \frac{2\cos\theta}{4\sin\theta} = \frac{1}{2}\cot\theta$

$(-2, 3+\sqrt{3}) \Rightarrow \quad x = -4\cos\theta = -2$

$\cos\theta = 1/2 \Rightarrow \theta = \pi/3,\ \theta = 5\pi/3$

$y = 3 + 2\sin\theta = 3 + \sqrt{3}$

$\sin\theta = \sqrt{3}/2 \Rightarrow \theta = \pi/3,\ \theta = 5\pi/3$

$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{1/2}}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

$$\boxed{y - 3 - \sqrt{3} = \frac{2}{\sqrt{3}}(x+2)}$$

9. $\frac{dy}{dx} = \frac{2t}{-1} = -2t$

$\boxed{\begin{array}{l}\underline{\text{Horizontal}}: (2,0) \\ \text{No } \underline{\text{Vertical}}\end{array}}$

$t=0 \Rightarrow x=2, y=0$

10. $\frac{dy}{dx} = \frac{2t+3}{1} = 2t+3$

$\boxed{\begin{array}{l}\underline{\text{Horizondal}}: (5/2, -9/4) \\ \text{No Vertical}\end{array}}$

$t=-3/2 \Rightarrow x = \frac{5}{2}, y = -9/4$

11. $\frac{dy}{dx} = \frac{4\cancel{\cos\theta}}{-16\cancel{\cos\theta}\sin\theta} = -\frac{1}{4\sin\theta}$

$\boxed{\begin{array}{l}\underline{\text{No Horizontal}} \\ \text{Vertical}: (8,0)\end{array}}$

$\sin\theta = 0 \Rightarrow \theta = n\pi$, n-intg.

$x = 8\cos^2(n\pi) = 8$

$y = 4\sin(n\pi) = 0$

12. $\frac{dy}{dx} = \frac{2\sin\theta}{2} = \sin\theta$

$\sin\theta = 0 \Rightarrow \theta = n\pi$, n-intg.

$x = 2n\pi$

$y = \begin{cases} 0 & ,n \text{ even} \\ 4 & ,n \text{ odd} \end{cases}$

$\boxed{\begin{array}{l} \underline{\text{Horizontal}}: (2n\pi, 0) \quad n\text{-even} \\ \quad\quad \text{or} \\ \quad\quad (2n\pi, 4) \quad n\text{-odd} \\ \underline{\text{No Vertical}} \end{array}}$

13. $\frac{dy}{dt} = \frac{3t^2-3}{2t+1}$

$\boxed{\begin{array}{l} \underline{\text{Horizontal}}: (0,-2), (-2,2) \\ \underline{\text{Vertical}}: (-9/4, 11/8) \end{array}}$

$3t^2-3=0 \Rightarrow t=\pm 1$ ↗ $x=0, y=-2$ ↘ $x=-2, y=2$

$2t+1=0 \Rightarrow t=-1/2 \rightarrow x=-\frac{9}{4}, y=11/8$

14. $\frac{dy}{dx} = \frac{4\cos(2\theta)}{-2\sin\theta} = -2\frac{\cos(2\theta)}{\sin\theta}$

$\boxed{\begin{array}{l} \underline{\text{Horizontal}}: \left(2\cos\left(\pi/4 + \pi/2\, n\right), 2(-1)^n\right) \\ \quad n\text{-ing.} \\ \underline{\text{Vertical}}: (2(-1)^n, 0) \end{array}}$

$\cos(2\theta)=0 \Rightarrow 2\theta = \frac{\pi}{2} + \pi n$ $\quad y = 2\sin\left(\frac{\pi}{2}+\pi n\right)$

$\theta = \pi/4 + \frac{\pi}{2}n$ $\quad y = 2(-1)^n$

$x = 2\cos\left(\pi/4 + \pi/2\, n\right)$

$\sin\theta = 0 \Rightarrow \theta = n\pi \rightarrow y = \sin(2n\pi) = 0$

$x = 2\cos(n\pi) = 2(-1)^n$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

15. $\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 12t^2$

$$L = \int_{-1}^{1} \sqrt{4t^2 + 144t^4}\, dt = 2\int_0^1 \sqrt{4t^2 + 144t^4}\, dt$$

$u = 1 + 9t^2$
$du = 18t\, dt$

$$= \frac{2}{9}\int_1^{10} u^{1/2}\, du = \frac{2}{9}\cdot\frac{2}{3}u^{3/2}\Big]_1^{10} = \boxed{\frac{4}{27}(10)^{3/2}}$$

16. $\frac{dx}{dt} = \frac{1}{2\sqrt{t}} \qquad \frac{dy}{dt} = 3$

$$L = \int_1^2 \sqrt{\frac{1}{4t} + 9}\, dt \approx \boxed{3.062}$$

17. $\frac{dx}{dt} = -e^{-t}\cos t - e^{-t}\sin t = -e^{-t}(\cos t + \sin t) = \frac{\cos t + \sin t}{-e^t}$

$\frac{dy}{dt} = -e^{-t}\sin t + e^{-t}\cos t = \frac{\cos t - \sin t}{e^t}$

$$L = \int_0^{\pi/2} \sqrt{\frac{(\cos t + \sin t)^2}{e^{2t}} + \frac{(\cos^2 - \sin^2 t)^2}{e^{2t}}}\, dt = \int_0^{\pi/2} \sqrt{\frac{2}{e^{2t}}}\, dt$$

$$L = \sqrt{2}\int_0^{\pi/2} e^{-t}\, dt = -\sqrt{2}\, e^{-t}\Big]_0^{\pi/2} = -\sqrt{2}\left(e^{-\pi/2} - 1\right) = \boxed{\sqrt{2}\left(1 - \frac{1}{e^{\pi/2}}\right)}$$

18. $\frac{dx}{dt}=1 \qquad \frac{dy}{dt} = \frac{t^4}{2} - \frac{1}{2t^4} = \frac{t^8-1}{2t^4}$

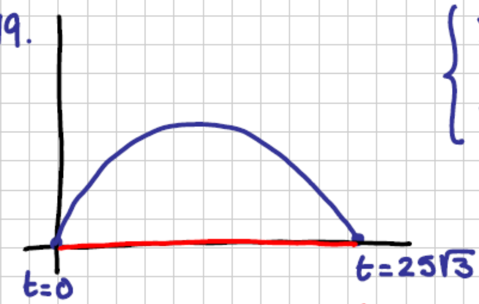
$$L=\int_1^2 \sqrt{1+\frac{(t^8-1)^2}{4t^8}}\,dt = \int_1^2 \sqrt{\frac{4t^8+t^{16}-2t^8+1}{4t^8}}\,dt =$$

$$=\int_1^2 \sqrt{\frac{t^{16}+2t^8+1}{4t^8}}\,dt = \int_1^2 \sqrt{\frac{(t^8+1)^2}{4t^8}}\,dt = \int_1^2 \frac{t^8+1}{2t^4}\,dt$$

$$=\frac{1}{2}\int_1^2 t^4+t^{-4}\,dt = \frac{1}{2}\left(\frac{t^5}{5}-\frac{1}{3}t^{-3}\Big]_1^2\right)$$

$$=\frac{1}{2}\left(\frac{32}{5}-\frac{1}{24}-\frac{1}{5}+\frac{1}{3}\right) = \frac{1}{2}\left(\frac{31}{5}+\frac{7}{24}\right) = \frac{31}{10}+\frac{7}{48} = \boxed{\frac{779}{240}}$$

19.



$$\begin{cases} x = 50t \\ y = 50\sqrt{3}t - 16t^2 = -2t(t-25\sqrt{3}) \end{cases}$$

Possible Range: $x = 50(25\sqrt{3})$

$\longrightarrow$ $x = 1250\sqrt{3} \approx \boxed{2{,}165.064 \text{ ft}}$

$$L=\int_0^{25\sqrt{3}} \sqrt{50^2+(50\sqrt{3}-32t)^2}\,dt \approx \boxed{26{,}727.803 \text{ ft}}$$

20.

$$\frac{dx}{dt} = \frac{4(1+t^3) - 4t(3t^2)}{(1+t^3)^2}$$

$$\frac{dx}{dt} = \frac{4-8t^3}{(1+t^3)^2}$$

a) $$\frac{dy}{dt} = \frac{8t(1+t^3) - 4t^2(3t^2)}{(1+t^3)^2}$$

$$\frac{dy}{dt} = \frac{8t + 8t^4 - 12t^4}{(1+t^3)^2} = \frac{-4t^4 + 8t}{(1+t^3)^2}$$

$$\frac{dy}{dt} = 0 \Rightarrow \boxed{t = 0\,,\ t = \sqrt[3]{2}}$$

b) $$L = \int_0^{\sqrt[3]{2}} \sqrt{\frac{(4-8t^3)^2 + (-4t^4+8t)^2}{(1+t^3)^4}}\, dt$$