Polar Coordinates

Throughout this course, we have denoted a point in the plane by an ordered pair (x, y), where the numbers x and y denote the directed (i.e., signed positive or negative) distance between the point and each of two perpendicular lines, the x-axis and the y-axis. The elements of this ordered pair are called *coordinates*, and the coordinates used in this particular method of identifying points in the plane are called *Cartesian coordinates*.

In this lecture, we introduce an alternative coordinate system known as the *polar coordinate system*. In this system, a point in the plane is identified by an ordered pair (r, θ), where:

- r is the directed distance from a point designated as the *pole*, and
- θ is the angle, in radians, that a ray between the pole and the point makes with a ray designated as the *polar axis*.

The coordinates r and θ are called *polar coodinates*.

The pole is the point $(0, 0)$ in Cartesian coordinates, and has polar coordinates $(0, \theta)$ for *any* value of θ. The polar axis corresponds to the positive x-axis. An angle θ is considered positive if measured in the counterclockwise direction from the polar axis, and negative if measured in the clockwise direction.

Using these conventions, the Cartesian coordinates of a point can easily be obtained from the polar coordinates using the relations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Since $\sin\theta$ and $\cos\theta$ are not one-to-one, and since r is allowed to assume negative values, it follows that each point in the plane has infinitely many representations in polar coordinates.

Example Compute the Cartesian coordinates of the following points whose polar coordinates are given.

1. $(1, \pi/4)$
2. $(-1, 5\pi/4)$
3. $(1, 9\pi/4)$

Solution Using the relations

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we have:

1. $x = 1 \cdot \cos(\pi/4) = \sqrt{2}/2$, $y = 1 \cdot \sin(\pi/4) = \sqrt{2}2$
2. $x = -\cos(5\pi/4) = -(-\sqrt{2}/2) = \sqrt{2}/2$, $y = -\sin(\pi/4) = -(-\sqrt{2}2) = \sqrt{2}/2$
3. $x = 1 \cdot \cos(9\pi/4) = \cos(\pi/4) = \sqrt{2}/2$, $y = 1 \cdot \sin(9\pi/4) = \sin(\pi/4) = \sqrt{2}/2$

□

The polar coordinates of a point can be obtained from the Cartesian coordinates as follows:

$$r = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

It should be noted that because $\tan\theta$ is not one-to-one on the interval $0 \leq \theta < 2\pi$, it is necessary to consider the signs of x and y in order to make sure that the proper value of θ is used to represent the point (x, y). Otherwise, the point (r, θ) may lie in the wrong quadrant of the plane.

Example Compute the polar coordinates of the following points whose Cartesian coordinates are given.

1. $(-\sqrt{3}/2, 1/2)$
2. $(-1, -1)$

Solution Using the relations

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x},$$

we have:

1. $$r^2 = (-\sqrt{3}/2)^2 + (1/2)^2 = 3/4 + 1/4 = 1, \quad \tan\theta = -\frac{1}{\sqrt{3}}.$$

   It follows that $r = 1$. Because the x-coordinate of the point is negative, we should seek a value of θ that lies in the interval $(\pi/2, 3\pi/2)$. However, the range of the inverse tangent function lies in the interval $(-\pi/2, \pi/2)$, and therefore $\theta = \tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Since tangent has a *period* of π, it follows that

   $$\tan(\theta + \pi) = \tan\theta$$

   for any θ; in other words, its values repeat after every π units. Since

   $$\tan(5\pi/6) = \tan(-\pi/6 + \pi) = \tan(-\pi/6) = -\frac{1}{\sqrt{3}},$$

it follows that $\theta = 5\pi/6$ satisfies the relation $\tan\theta = y/x$, and since $5\pi/6$ lies in the interval $(\pi/2, 3\pi/2)$, it is a correct value of θ for this point.

2.
$$r^2 = (-1)^2 + (-1)^2 = 2, \quad \tan\theta = \frac{-1}{-1} = 1.$$

It follows that $r = \sqrt{2}$. Because the x-coordinate of the point is negative, we should seek a value of θ that lies in the interval $(\pi/2, 3\pi/2)$. However, we have $\tan^{-1}(1) = \pi/4$, which is not in that interval. Since

$$\tan(5\pi/4) = \tan(\pi/4 + \pi) = \tan(\pi/4) = 1,$$

it follows that $\theta = 5\pi/4$ satisfies the relation $\tan\theta = y/x$, and since $5\pi/4$ lies in the interval $(\pi/2, 3\pi/2)$, it is a correct value of θ for this point.

□

A *polar equation* is an equation of the form $r = f(\theta)$. Such an equation defines a curve in the plane by assigning a distance from the pole to each angle θ via the function $f(\theta)$. For example, the simple polar equation $r = k$, where k is a constant, describes a circle of radius k. The *graph* of a polar equation is the set of all points in the plane that can be described using polar coordinates that satisfy the equation. This definition is worded as such in order to take into account that each point in the plane can have infinitely many representations in polar coordinates.

Example Sketch the curve described by the polar equation

$$r = \cos 2\theta, \quad 0 \leq \theta \leq 2\pi.$$

Solution This curve can be sketched by evaluating $r = \cos 2\theta$ at several values of θ. For each such value, the point $(r, \theta) = (\cos 2\theta, \theta)$ can be plotted by traversing r units along the ray that makes the angle θ with the polar axis (which is the x-axis), if r is positive; otherwise, use the ray that makes the angle $\theta + \pi$ with the polar axis. The curve $r = \cos 2\theta$ is illustrated in Figure 1. The circles indicate the points corresponding to $\theta = 0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4$, and 2π. □

Example Sketch the curve described by the polar equation

$$r = \sin\theta, \quad 0 \leq \theta \leq 2\pi.$$

Solution Figure 2 displays the curve, which can be plotted using the same approach as in the previous example. The circles indicate the points corresponding to $\theta = 0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4$, and 2π. The circle is traced twice, once for $0 \leq \theta \leq \pi$, and again for $\pi \leq \theta \leq 2\pi$. □

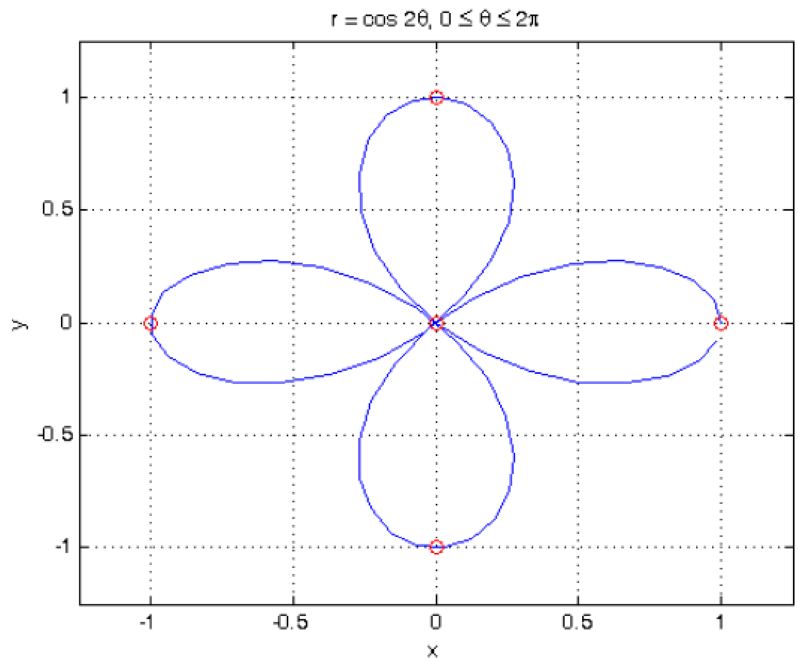

Figure 1: Curve described by the polar equation $r = \cos 2\theta$, where $0 \leq \theta \leq 2\pi$.

We now determine the slope of a tangent line of a polar curve. If the curve can be described by an equation of the form $y = F(x)$ for some differentiable function F, then, by the Chain Rule,

$$\frac{dy}{d\theta} = F'(x)\frac{dx}{d\theta},$$

but since $F'(x) = dy/dx$, it follows that

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Expressing x and y in polar coordinates and applying the Product Rule yields

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

It can be shown that this result also holds for curves that cannot be described by an equation of the form $y = F(x)$.

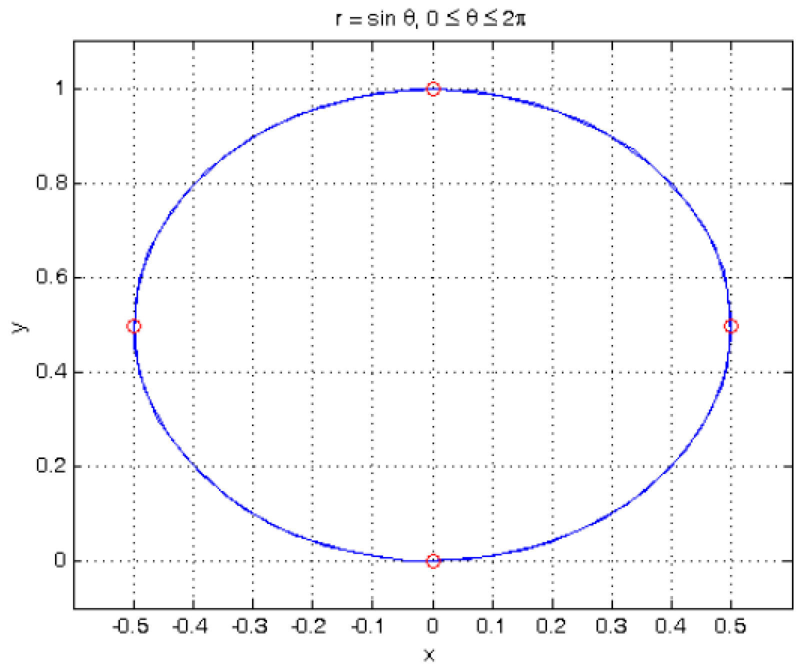

Figure 2: Curve described by the polar equation $r = \sin\theta$, where $0 \leq \theta \leq 2\pi$.

We make the following observations about tangents to polar curves, based on the above expression for their slope:

- Horizontal tangents occur when $dy/d\theta = 0$, provided that $dx/d\theta \neq 0$.
- Vertical tangents occur when $dx/d\theta = 0$, provided that $dy/d\theta \neq 0$.
- At the pole, when $r = 0$, the slope of the tangent is given by

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta}{\frac{dr}{d\theta}\cos\theta} = \tan\theta$$

  provided $dr/d\theta \neq 0$.

Example Given the curve defined by the polar equation $r = \sin\theta$, where $0 \leq \theta \leq \pi$, determine the values of θ at which the tangent to the curve is either horizontal or vertical.

Solution As we learned in the previous example, this curve is a circle with center $(0, 1/2)$ and radius $1/2$. The curve is displayed in Figure 2. Using the formula for the slope of the tangent, we have, by double-angle formulas,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ &= \frac{(\cos\theta)\sin\theta + (\sin\theta)\cos\theta}{(\cos\theta)\cos\theta - (\sin\theta)\sin\theta} \\ &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \\ &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= \tan 2\theta. \end{aligned}$$

Alternatively, we can use the relations

$$x = r\cos\theta, \quad y = r\sin\theta$$

to compute $dy/d\theta$ and $dx/d\theta$ directly. In this case, we have

$$x = \sin\theta\cos\theta, \quad y = \sin^2\theta$$

or, from double-angle and half-angle formulas,

$$x = \frac{\sin 2\theta}{2}, \quad y = \frac{1 - \cos 2\theta}{2}$$

and therefore

$$\frac{dy}{d\theta} = \sin 2\theta, \quad \frac{dx}{d\theta} = \cos 2\theta,$$

which yields $dy/dx = \tan 2\theta$ as before.

The tangent is horizontal when $dy/d\theta = 0$ and $dx/d\theta \neq 0$. This occurs when $\theta = 0, \pi/2$, or π. The tangent is vertical when $dx/d\theta = 0$ and $dy/d\theta \neq 0$. This occurs when $\theta = \pi/4$ and $\theta = 3\pi/4$. □

Summary

- A point can be represented by polar coordinates (r, θ), where r is the distance between the point and the origin, or pole, and θ is the angle that a line segment from the pole to the point makes with the positive x-axis.

- To convert from polar coordinates to Cartesian coordinates (x, y), one can use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

- To convert from Cartesian coordinates to polar coordinates, one can use $r = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1}(y/x)$ if $x < 0$. If $x < 0$, then $\theta = \tan^{-1}(y/x) + \pi$. If $x = 0$, $\theta = \pi/2$ if $y > 0$, and $-\pi/2$ if $y < 0$.

- To graph a curve defined by a polar equation of the form $r = f(\theta)$, one can compute r for various values of θ, and use polar coordinates to plot the corresponding points on the curve.

- To compute the slope of the tangent to a polar curve $r = f(\theta)$, one can differentiate $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$ with respect to θ, and then use the relation $dy/dx = (dy/d\theta)/(dx/d\theta)$.