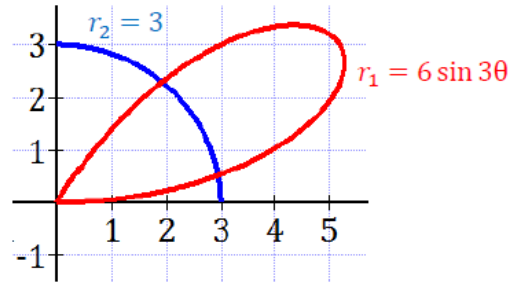
Derivatives and Equations in Polar Coordinates

1. The graphs of the polar curves $r_1 = 6\sin 3\theta$ and $r_2 = 3$ are shown to the right.
(You may use your calculator for all sections of this problem.)
   a) Find the coordinates of the points of intersection of both curves for $0 \le \theta < \frac{\pi}{2}$. Write your answers using polar coordinates.
   b) Write the coordinates of the points of intersection using now rectangular coordinates.



   c) Find $\left.\frac{dr_1}{d\theta}\right|_{\theta=\frac{\pi}{4}}$. Interpret the meaning of your answer in the context of the problem.
   d) For $0 \le \theta < \frac{\pi}{2}$, there are two points on r_1 with x-coordinate equal to 4. Find the subject points. Express your answer using polar coordinates.
   e) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r_1.
   f) Write in terms of x and y an equation for the line tangent to the graph of the curve r_1 at the point where $\theta = \frac{\pi}{4}$.

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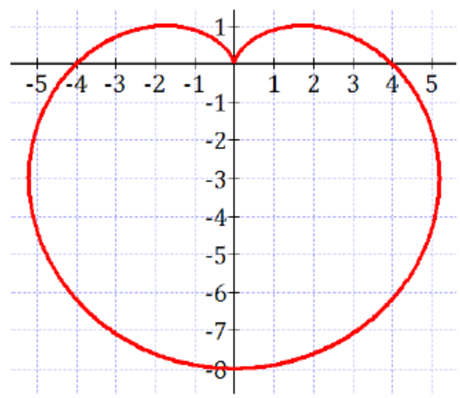
2. The graph of the polar curve $r = 4 - 4\sin\theta$ is shown to the right.
(You may use your calculator for all sections of this problem.)
   a) For $0 \le \theta < 2\pi$, there are two points on r with y-coordinate equal to -4. Find the subject points. Express your answers using polar coordinates.
   b) Write an expression for the x-coordinate of each point on the graph of $r = 4 - 4\sin\theta$. Express your answer in terms of θ.
   c) A particle moves along the polar curve $r = 4 - 4\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the time interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

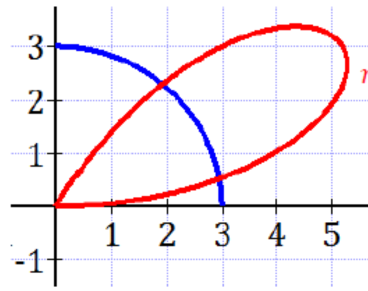


   d) Find $\left.\frac{dr}{dt}\right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.
   e) Find $\left.\frac{dx}{dt}\right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.

ANSWER KEY

Derivatives and Equations in Polar Coordinates

1. The graphs of the polar curves $r_1 = 6\sin 3\theta$ and $r_2 = 3$ are shown to the right.
(You may use your calculator for all sections of this problem.)



a) Find the coordinates of the points of intersection of both curves for $0 \le \theta < \frac{\pi}{2}$. Write your answers using polar coordinates.

Points of intersection are collision points:

$$6\sin 3\theta = 3 \to \theta = \frac{\pi}{18} \text{ and } \frac{5\pi}{18}$$

Or $\theta \approx 0.1745$ and 0.8726

$r = 3 \to (3, 0.1745)$ and $(3, 0.8726)$

b) Write the coordinates of the points of intersection using now rectangular coordinates.

$$(3, 0.1745) \to \begin{cases} x = r\cdot\cos\theta = 2.954 \\ y = r\cdot\sin\theta = 0.5209 \end{cases} \to (2.954, 0.5209)$$

$$(3, 0.8726) \to \begin{cases} x = r\cdot\cos\theta = 1.928 \\ y = r\cdot\sin\theta = 2.298 \end{cases} \to (1.928, 2.298)$$

c) Find $\left.\frac{dr_1}{d\theta}\right|_{\theta=\frac{\pi}{4}}$. Interpret the meaning of your answer in the context of the problem.

$$\text{By hand: } \frac{dr_1}{d\theta} = 18\cos 3\theta \to \left.\frac{dr_1}{d\theta}\right|_{\theta=\frac{\pi}{4}} = -9\sqrt{2}$$

$$\text{Using a calculator: } \left.\frac{d}{d\theta}(6\sin 3\theta)\right|_{\theta=\frac{\pi}{4}} \approx -12.7279$$

When the graph of $r_1 = 6\sin 3\theta$ is traced at $\theta = \frac{\pi}{4}$ radians the distance to the pole is decreasing at a rate equal to 12.7279 units per radian.

d) For $0 \le \theta < \frac{\pi}{2}$, there are two points on r_1 with x-coordinate equal to 4. Find the subject points. Express your answer using polar coordinates.

$$x = r_1\cdot\cos\theta = 6\sin 3\theta\cdot\cos\theta = 4 \to \theta \approx 0.253 \text{ and } 0.696$$

$$\theta \approx 0.253 \to r_1 = 6\sin(3(0.253)) = 4.1317 \to (4.137, 0.253)$$

$$\theta \approx 0.696 \to r_1 = 6\sin(3(0.696)) = 5.213 \to (5.213, 0.696)$$

e) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r_1.

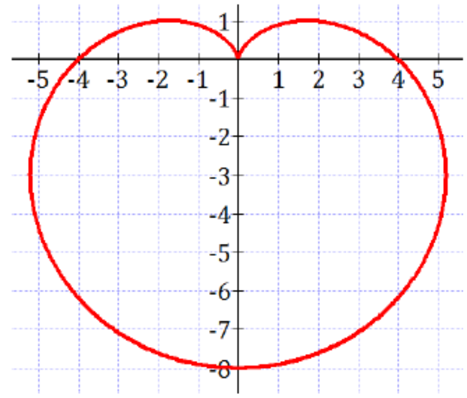
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta}$$

f) Write in terms of x and y an equation for the line tangent to the graph of the curve r_1 at the point where $\theta = \frac{\pi}{4}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{1}{2}$$

$$\left.\begin{aligned} x = r_1\cdot\cos\theta = 3 \\ y = r_1\cdot\sin\theta = 3 \end{aligned}\right\} \to y - 3 = \frac{1}{2}(x - 3)$$

2. The graph of the polar curve $r = 4 - 4\sin\theta$ is shown to the right.



(You may use your calculator for all sections of this problem.)

a) For $0 \le \theta < 2\pi$, there are two points on r with y-coordinate equal to -4. Find the subject points. Express your answers using polar coordinates.

$$y = r \cdot \sin\theta = (4 - 4\sin\theta)\sin\theta = -4$$
$$\rightarrow \theta \approx 3.8078 \quad \text{and} \quad 5.6169$$
$$\theta \approx 3.8078 \rightarrow r = 4 - 4\sin 3.8078 = 6.472$$
$$\rightarrow (6.472, 3.8078)$$
$$\theta \approx 5.6169 \rightarrow r = 4 - 4\sin 5.6169 = 6.472$$
$$\rightarrow (6.472, 5.6169)$$

b) Write an expression for the x-coordinate of each point on the graph of $r = 4 - 4\sin\theta$. Express your answer in terms of θ.

$$x = r \cdot \cos\theta = (4 - 4\sin\theta)\cos\theta$$

c) A particle moves along the polar curve $r = 4 - 4\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the time interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

$$x = (4 - 4\sin t^2)\cos t^2 = -1 \rightarrow t \approx 1.5536$$

d) Find $\left.\frac{dr}{dt}\right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.

$$r = 4 - 4\sin t^2$$

By hand: $\frac{dr}{dt} = -8t\cos t^2 \rightarrow \left.\frac{dr}{dt}\right|_{t=2} = -16\cos 4$

Using a calculator: $\left.\frac{d}{dt}(4 - 4\sin t^2)\right|_{t=2} \approx 10.458$

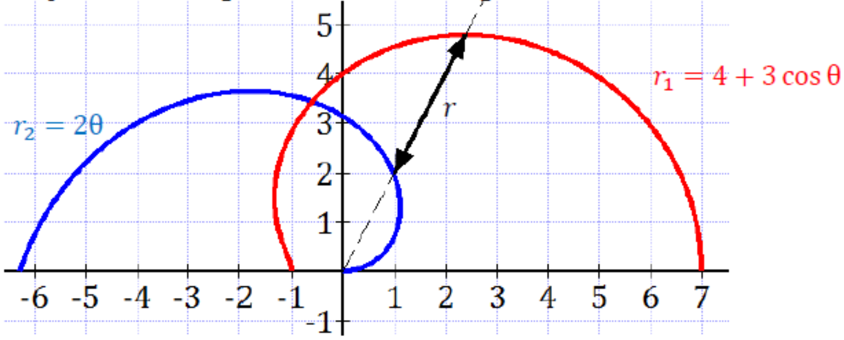
As the particle moves on the graph of $r = 4 - 4\sin\theta$, when $t = 2$ seconds the distance to the pole is increasing at a rate equal to 10.458 units per second.

e) Find $\left.\frac{dx}{dt}\right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.

Using a calculator: $\left.\frac{d}{dt}((4 - 4\sin t^2)\cos t^2)\right|_{t=2} \approx 14.4368$

As the particle moves on the graph of $r = 4 - 4\sin\theta$, when $t = 2$ seconds the particle moves to the right with a horizontal speed equal to 14.4368 units per second.

3. The graphs of the polar curves $r_1 = 4 + 3\cos\theta$ and $r_2 = 2\theta$ are shown below.



(Do NOT use your calculator for this problem unless indicated!)

a) Find the coordinates of the point of intersection of both curves for $0 \le \theta < \pi$. Write your answer using polar coordinates. *(You may use your calculator for this section.)*

b) As the curves are traced, the distance between them, $r(\theta)$, changes (see drawing.) Find an expression for $r(\theta)$ the distance between both curves in the interval $0 \le \theta \le \frac{\pi}{2}$.

c) Write in terms of θ an expression for $\frac{dr}{d\theta}$. Use your answer to find $\left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{3}}$. Interpret the meaning of your answer in the context of the problem.

d) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r_2.

e) Find the coordinates of the point where curve r_2 has a horizontal tangent line in the interval $0 < \theta < \pi$. Write your answer using rectangular coordinates. *(You may use your calculator for this section.)*

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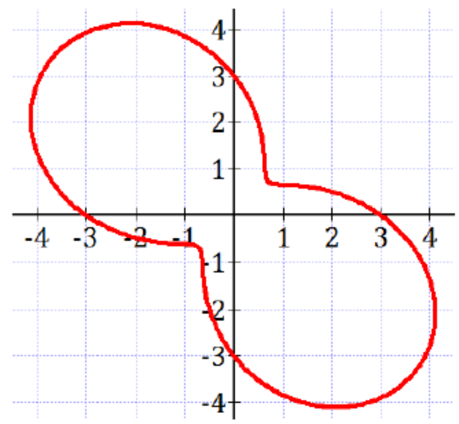
4. The graph of the polar curve $r = 3 - 2\sin(2\theta)$ for $0 \le \theta < 2\pi$ is shown to the right.
(You may use your calculator for all sections of this problem.)



a) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r.

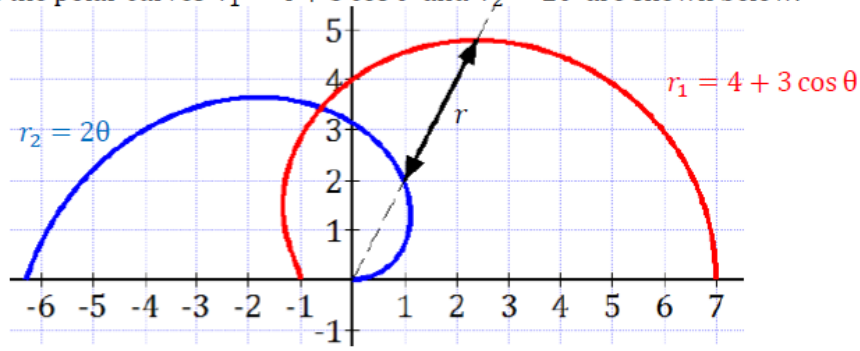
b) Find the coordinates of the point where curve r has a vertical tangent line in the interval $0 \le \theta < \pi$. Write your answer using polar coordinates.

c) Write in terms of x and y an equation for the line tangent to the graph of the curve r at the point where $\theta = \frac{\pi}{6}$.

d) A particle moves along the polar curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 2$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$. Interpret the meaning of your answer in the context of the problem.

e) Assume now that for the particle whose motion was described in section (d) we have $\theta = 2t$. Find the position vector of the particle $\langle x(t), y(t)\rangle$ in terms of t. Use your calculator to find the velocity vector and the speed of the particle at $t = 1.5$.

3. The graphs of the polar curves $r_1 = 4 + 3\cos\theta$ and $r_2 = 2\theta$ are shown below.



(Do NOT use your calculator for this problem unless indicated!)

a) Find the coordinates of the point of intersection of both curves for $0 \le \theta < \pi$. Write your answer using polar coordinates. *(You may use your calculator for this section.)*

The point of intersection is also a collision point:

$$4 + 3\cos\theta = 2\theta \to \theta \approx 1.7429$$

$$r = 2(1.7429) = 3.4859 \to (3.4859, 1.7429)$$

b) As the curves are traced, the distance between them, $r(\theta)$, changes (see drawing.) Find an expression for $r(\theta)$ the distance between both curves in the interval $0 \le \theta \le \frac{\pi}{2}$.

$$r(\theta) = r_1 - r_2 = 4 + 3\cos\theta - 2\theta$$

c) Write in terms of θ an expression for $\frac{dr}{d\theta}$. Use your answer to find $\left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{3}}$. Interpret the meaning of your answer in the context of the problem.

$$\frac{dr}{d\theta} = -3\sin\theta - 2 \to \left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{3}} = -\frac{3\sqrt{3}}{2} - 2$$

When the graphs of $r_1 = 6\sin 3\theta$ and $r_2 = 2\theta$ are traced at $\theta = \frac{\pi}{3}$ radians the distance between the two graphs is decreasing at a rate equal to $-\frac{3\sqrt{3}}{2} - 2$ units per radian.

d) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r_2.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}$$

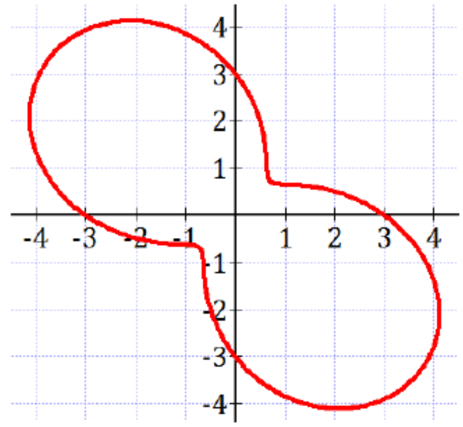
e) Find the coordinates of the point where curve r_2 has a horizontal tangent line in the interval $0 < \theta < \pi$. Write your answer using rectangular coordinates. *(You may use your calculator for this section.)*

$$\frac{dy}{dx} = 0 \to \sin\theta + \theta\cos\theta = 0 \to \theta \approx 2.0287$$

$$r = 2(2.0287) = 4.0575$$

$$\begin{cases} x = r \cdot \cos\theta = -1.7939 \\ y = r \cdot \sin\theta = 3.639 \end{cases} \to (-1.7939, 3.639)$$

4. The graph of the polar curve $r = 3 - 2\sin(2\theta)$ for $0 \le \theta < 2\pi$ is shown to the right.
(You may use your calculator for all sections of this problem.)



a) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-4\cos 2\theta \sin\theta + (3 - 2\sin(2\theta))\cos\theta}{-4\cos 2\theta\cos\theta - (3 - 2\sin(2\theta))\sin\theta}$$

b) Find the coordinates of the point where curve r has a vertical tangent line in the interval $0 \le \theta < \pi$. Write your answer using polar coordinates.

$$\frac{dy}{dx} \text{ is undefined} \to -4\cos 2\theta\cos\theta - (3 - 2\sin(2\theta))\sin\theta = 0 \to \theta \approx 2.670$$

$$r = 3 - 2\sin\big(2(2.670)\big) = 4.6177 \to (4.6177, 2.670)$$

c) Write in terms of x and y an equation for the line tangent to the graph of the curve r at the point where $\theta = \frac{\pi}{6}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} \approx -0.041 \qquad \left(\text{by hand: } \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{3\sqrt{3}-5}{-3-\sqrt{3}}\right)$$

$$\left.\begin{aligned} x &= r\cdot\cos\theta = \frac{3\sqrt{3}-3}{2} = 1.098 \\ y &= r\cdot\sin\theta = \frac{3-\sqrt{3}}{2} = 0.6339 \end{aligned}\right\} \to y - 0.6339 = -0.041(x - 1.098)$$

d) A particle moves along the polar curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 2$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$. Interpret the meaning of your answer in the context of the problem.

$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = (-4\cos(2\theta))(2)$$

$$\left.\frac{dr}{dt}\right|_{\theta=\frac{\pi}{6}} = -4$$

As the particle moves on the graph of $r = 3 - 2\sin(2\theta)$, when it is at the point where $\theta = \frac{\pi}{6}$ radians the distance to the pole is decreasing at a rate equal to 4 units per second.

e) Assume now that for the particle whose motion was described in section (d) we have $\theta = 2t$. Find the position vector of the particle $\langle x(t), y(t)\rangle$ in terms of t. Use your calculator to find the velocity vector and the speed of the particle at $t = 1.5$.

$$\left.\begin{aligned} x &= r\cdot\cos\theta = (3 - 2\sin(4t))\cos(2t) \\ y &= r\cdot\sin\theta = (3 - 2\sin(4t))\sin(2t) \end{aligned}\right\} \to \langle (3 - 2\sin(4t))\cos(2t), (3 - 2\sin(4t))\sin(2t)\rangle$$

$$\text{Velocity vector: } \left\langle \left.\frac{dx}{dt}\right|_{t=1.5}, \left.\frac{dy}{dt}\right|_{t=1.5} \right\rangle = \langle 6.600, -8.130\rangle$$

$$\text{Speed: } \sqrt{\left(\left.\frac{dx}{dt}\right|_{t=1.5}\right)^2 + \left(\left.\frac{dy}{dt}\right|_{t=1.5}\right)^2} = 10.472$$