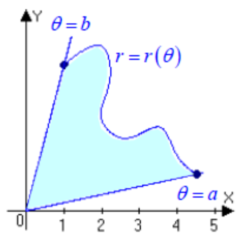
Geometric quantities for polar curves

What you need to know already:	*What you can learn here:*
➤ How to use integrals to compute areas and lengths of regions bounded by regular and parametric curves.	➤ How to use integrals to compute the same quantities for a region bounded by one or more polar curves.

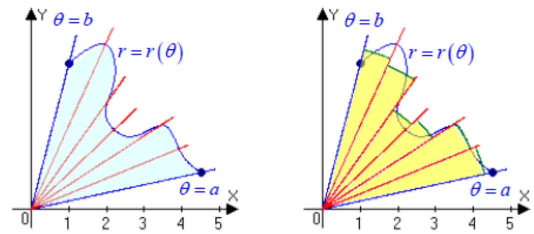
In the applications we have seen so far, we have used slices that were vertical or horizontal. Notice that this makes full use of the Cartesian coordinates we normally use. But what if we are given information in polar coordinates?

Polar coordinates use a radically different approach to identify. Therefore it should not surprise you that the method needed to compute areas of regions bounded by polar curves is a substantial variation of those we have used so far.

To develop such method, we need to use, once again, the four step process to construct an integral, but in a novel way. Since in polar coordinates the independent variable θ traces the polar curve radially, by rotating around the pole counter clockwise, let us assume that the region of interest is bounded by a polar curve $r = r(\theta)$ and two radii at $\theta = a$ and $\theta = b$, and as shown here.



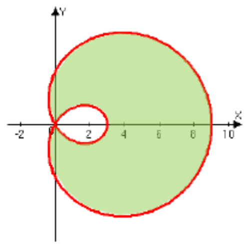
Let us now slice this region like a piece of pie centered at the pole, through several more radii. In this way we can approximate the area of each slice by using a circular sector whose angle is a small $\Delta\theta$ and whose radius is the distance from the pole to the curve at some point within the slice.



Notice that each approximating slice looks like a triangle, but it isn't! Since we work in polar coordinates, we have to rotate around the pole, so that what you are seeing is a circular sector.

Example: $r=3(1+2\cos\theta)$

What about the area of the region between the outer and inner loops of the same polar curve? All we need to do is find the area bounded by the *outer* loop and subtract the area of the *inner* loop from it.



For the outer loop, we can use symmetry, thus looking at twice the area of the top part, or simply follow the rotation. We see that such loop starts at $\theta=-\frac{2\pi}{3}$ and ends at $\theta=\frac{2\pi}{3}$. Therefore the area bounded by the outer loop is:

$$A_{outer}=\frac{1}{2}\int_{-2\pi/3}^{2\pi/3}9(1+2\cos\theta)^2\,d\theta$$

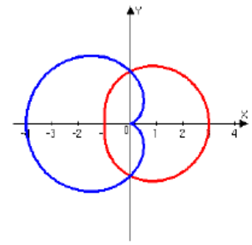
This can be computed as before and then we need to subtract the inner loop, leading to:

$$A_{outer}=A_{outer}-A_{inner}=$$

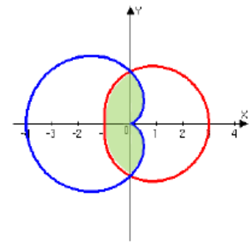
$$=\frac{1}{2}\int_{-2\pi/3}^{2\pi/3}9(1+2\cos\theta)^2\,d\theta-\frac{1}{2}\int_{2\pi/3}^{4\pi/3}9(1+2\cos\theta)^2\,d\theta$$

Example: $r_1=2+\cos\theta$, $r_2=2-2\cos\theta$

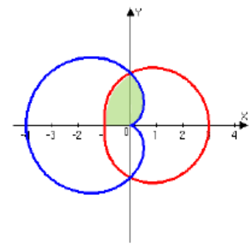
By graphing these curves we can see several regions.



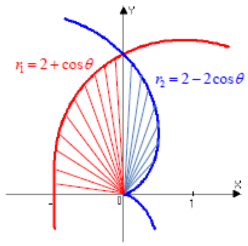
Let us compute the area of the region inside both curves.



This region is symmetric, so we can consider only the portion above the *x*-axis.



Moreover, by rotating counter clockwise, we notice that this region consists of two smaller regions:



In the first quadrant, for $0 \le \theta \le \frac{\pi}{2}$, we use only the second curve, while in the second quadrant, for $\frac{\pi}{2} \le \theta \le \pi$, we use only the first curve.

Therefore the desired formula is:

$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} (2 - 2\cos\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/2}^{\pi} (2 + \cos\theta)^2\, d\theta\right]$$

$$= \int_0^{\pi/2} (2 - 2\cos\theta)^2\, d\theta + \int_{\pi/2}^{\pi} (2 + \cos\theta)^2\, d\theta$$

This is a basic integral whose computation I leave to you.

That looks confusing! I am afraid to ask what happens when we tackle lengths, surface areas and volumes.

Well, cheer up: although surface areas and volumes can be tackled when dealing with polar curves, they lead to very messy formulae that are beyond the scope of our curse. I meant course! So, we are left with arc length, which turns out to be surprisingly simple.

Technical fact

If $r = r(\theta),\ a \le \theta \le b$ is a finite ***polar*** curve, then its ***length*** is given by:

$$L = \int_a^b \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Proof

We start by writing the polar curve in parametric form:

$$x = r\cos\theta \quad \Rightarrow \quad x' = r'\cos\theta - r\sin\theta$$
$$y = r\sin\theta \quad \Rightarrow \quad y' = r'\sin\theta + r\cos\theta$$

Now we use the formula for the parametric arc length:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, d\theta =$$

$$= \int_a^b \sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2}\, d\theta$$

By expanding the two squares, the two double products cancel each other and the remaining terms can be grouped as follows:

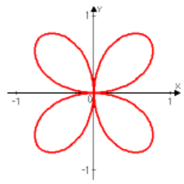
$$L = \int_a^b \sqrt{(r)^2\left(\sin^2\theta + \cos^2\theta\right) + (r')^2\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta$$

But what we have in brackets now is one side of the basic Pythagorean identity, whose other side is 1. Therefore, we can write the formula as:

$$L = \int_a^b \sqrt{(r)^2 + (r')^2}\, d\theta$$

as claimed.

Example: $r = \sin 2\theta$



The length of one petal of this rose is given by:

$$L = \int_0^{\pi/2} \sqrt{\sin^2 2\theta + 4\cos^2 2\theta}\, d\theta$$

You may want to try and compute this integral. Once again, we get a tough one, but despair not: a method for handling these difficult definite integrals is coming.

Just as with this example, other examples tend to be equally challenging, so the issue will be only to set up the integral, which boils down to identifying the limits of integration.

Summary

- The formula for the area of a region bounded by a polar curve is obtained by using the four step process to construct integrals and is based on the area of a circular sector, rather than a rectangle.
- The main difficulty in setting up the required integrals is in identifying the polar region, or sometimes regions, involved.
- The formula for arc length of a polar curve is obtained algebraically from the parametric version.
- General formulae for surface areas and volumes tend to be complicated enough to be deferred to a later course.

Common errors to avoid

- Don't rush over the identification of the polar region: it is the key element and it can be tricky!
- Remember that you are dealing with polar curves, so don't trace the curve left to right, but counter clockwise.

Learning questions for Section I 5-10

Review questions:

1. Explain how to set up the integral representing the area of a region bounded by polar curves.

2. Explain how to set up the integral representing the length of a polar curve.

Memory questions:

1. What geometrical shapes are used to construct the area formula in polar coordinates?

2. What is the formula for the area bounded by a polar curve of the form $r = f(\theta),\ \alpha \le \theta \le \beta$?

3. What is the integral formula for the arc length of the graph of a polar function $r = r(\theta)$?

4. When we analyze a polar region to identify the integration limits, how do we scan the region?

Computation questions:

In questions 1-16, set up the integral that provides the area of the region described there. If possible, compute the integral.

1. The region bounded by the first "*loop*" of the polar curve $r = \sqrt{\theta}e^{\theta/2}$ and the polar axis.

2. The region below the polar axis and contained between the two loops of the curve $r = 2 - 6\sin\theta$.

3. The region bounded by $y = |x|$ and $r = 3 - 2\sin\theta$.

4. The region inside the circle $r = -6\cos\theta$ and outside the cardioid $r = 2 - 2\cos\theta$.

5. The region enclosed by one loop of the curve $r = 1 + \sin 3\theta$.

6. The region common to the circles $r = \sin\theta$ and $r = \cos\theta$.

7. The figure-8 region bounded by both $r = 1 - \cos\theta$ and $r = 1 + \cos\theta$

8. The finite region bounded by the loop in the *conchoid* $r = 4 + 2\sec\theta$.

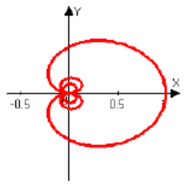
9. The region of intersection between the two circles $r = 2\sin\theta$ and $r = 1$.

10. The finite region that is inside both the rose $r = 2\sin 4\theta$ and the circle $r = 1$.

11. The region inside both $r = 2\sin\theta$ and $r = 3 - 2\sin\theta$.

12. The bib determined by the polar curve $r = 1 - 2\sin\theta$.

13. The region bounded by the outermost layer of the *cochleoid* $r = \dfrac{\sin\theta}{\theta}$ shown here:



14. One petal of the rose $r = 4\cos 3\theta$

15. The region inside $r = -6\cos\theta$ and outside $r = 2 - 2\cos\theta$

16. The region outside $r = -6\cos\theta$ and inside $r = 2 - 2\cos\theta$ and the region common to them.

In questions 17-21 ,set up the integral that provides the length of the polar curve described. If possible, compute the integral.

17. The first loop of the spiral $r = e^{\theta/2}$

18. One loop of the rose $r = 2\cos 3\theta$

19. The rabbit ears $r = \sin(\pi\cos\theta)$

20. The limaçon $r = 1 + 2\sin\theta$

21. $r = 3 - 2\sin\theta$

22. Which function of n represents the length of the first n "*loops*" of the spiral $r = \theta$? I expect you to evaluate any integrals that may be needed for the construction of such function.

Theory questions:

1. In the formula for the area bounded by a polar curve, is it important that the limits be used in increasing order?

2. Is it possible for two polar regions to intersect at a point that does NOT corresponds to the same value of θ for both curves?

3. Given the polar arc length formula, what would be the formula for the surface area obtained by rotating a polar curve around the x axis?

4. In the formula for the area bounded by a polar curve, which θ value goes in the bottom of the integral and which on top?

5. Which formula is used to simplify the expression in the formula for the arc length of a polar curve?

What questions do you have for your instructor?