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Sequences and Series

Consider the following sum:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^i} + \cdots$$

The dots at the end indicate that the sum goes on forever. Does this make sense? Can we assign a numerical value to an infinite sum? While at first it may seem difficult or impossible, we have certainly done something similar when we talked about one quantity getting "closer and closer" to a fixed quantity. Here we could ask whether, as we add more and more terms, the sum gets closer and closer to some fixed value. That is, look at

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{2} \\
\frac{3}{4} &= \frac{1}{2} + \frac{1}{4} \\
\frac{7}{8} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\
\frac{15}{16} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}
\end{aligned}$$

and so on, and ask whether these values have a limit. It seems pretty clear that they do, namely 1. In fact, as we will see, it's not hard to show that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^i} = \frac{2^i - 1}{2^i} = 1 - \frac{1}{2^i}$$

and then

$$\lim_{i\to\infty} 1 - \frac{1}{2^i} = 1 - 0 = 1.$$

There is one place that you have long accepted this notion of infinite sum without really thinking of it as a sum:

$$0.3333\bar{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \frac{1}{3},$$

for example, or

$$3.14159\ldots = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \cdots = \pi.$$

Our first task, then, to investigate infinite sums, called **series**, is to investigate limits of **sequences** of numbers. That is, we officially call

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^i} + \cdots$$

a series, while

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots, \frac{2^i - 1}{2^i}, \ldots$$

is a sequence, and

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \lim_{i\to\infty} \frac{2^i - 1}{2^i},$$

that is, the value of a series is the limit of a particular sequence.