The Ratio Test

Let $\sum u_k$ be a series with positive terms and suppose

$$\lim_{k \to \infty} \frac{u_{k+1}}{u_k} = \rho$$

a) If $\rho < 1$, the series converges.

b) If $\rho > 1$ or $\rho = \infty$, the series diverges.

c) If $\rho = 1$, the series may converge or diverge.

Example 1

Use the Ratio Test to determine whether

$$\sum_{k=1}^{\infty} \frac{1}{k!}$$

converges or diverges.

<u>Solution</u>:

$$u_k = \frac{1}{k!} \quad \text{and} \quad u_{k+1} = \frac{1}{(k+1)!}$$

Example 1 (continued)

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$$

$$= \lim_{k\to\infty} \frac{\left(\frac{1}{(k+1)!}\right)}{\left(\frac{1}{k!}\right)} = \lim_{k\to\infty} \frac{1}{(k+1)!} \cdot \frac{k!}{1}$$

$$= \lim_{k\to\infty} \frac{k \cdot (k-1) \cdot (k-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1}{(k+1) \cdot k \cdot (k-1) \cdot (k-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1}$$

$$= \lim_{k\to\infty} \frac{1}{(k+1)} = 0 < 1$$

So by the Ratio Test, the series converges.

Example 2

Use the Ratio Test to determine whether

$$\sum_{k=1}^{\infty} \frac{k^k}{k!}$$

converges or diverges.

<u>Solution</u>:

$$u_k = \frac{k^k}{k!} \quad \text{and} \quad u_{k+1} = \frac{(k+1)^{k+1}}{(k+1)!}$$

Example 2 (continued)

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$$

$$= \lim_{k\to\infty} \frac{\left(\frac{(k+1)^{k+1}}{(k+1)!}\right)}{\left(\frac{k^k}{k!}\right)} = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k}$$

$$= \lim_{k\to\infty} \frac{(k+1)^{k+1} \cdot k!}{k^k \cdot (k+1) \cdot k!}$$

$$= \lim_{k\to\infty} \frac{(k+1)^{k+1}}{k^k \cdot (k+1)}$$

Example 2 (continued)

$$\rho = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{k^k \cdot (k+1)} = \lim_{k\to\infty} \frac{(k+1) \cdot (k+1)^k}{k^k \cdot (k+1)}$$

$$= \lim_{k\to\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k$$

$$= \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e > 1$$

So by the Ratio Test, the series diverges.

Example 3

Determine whether

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} + \cdots$$

converges or diverges.

Solution:

$$u_k = \frac{1}{2k-1} \text{ and } u_{k+1} = \frac{1}{2(k+1)-1}$$

Example 3 (continued)

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$$

$$= \lim_{k\to\infty} \frac{\left(\frac{1}{2(k+1)-1}\right)}{\left(\frac{1}{2k-1}\right)} = \lim_{k\to\infty} \frac{1}{2(k+1)-1} \cdot \frac{2k-1}{1}$$

$$= \lim_{k\to\infty} \frac{2k-1}{2k+1} = 1$$

So the Ratio Test doesn't tell us if the series converges or diverges!

When this happens, you need to use a different convergence test.

Example 3 (continued)

Integral Test:

$$\int_1^\infty \frac{1}{2x-1}\,dx = \lim_{l\to\infty}\int_1^l \frac{1}{2x-1}\,dx$$
$$= \lim_{l\to\infty}\frac{1}{2}\ln(2x-1)\Big|_1^l$$
$$= \lim_{l\to\infty}\left(\frac{1}{2}\ln(2l-1)-\frac{1}{2}\ln(2\cdot 1-1)\right)$$
$$= \infty$$

By the Integral Test, since $\int_1^\infty \frac{1}{2x-1}\,dx$ diverges, $\sum_{k=1}^\infty \frac{1}{2k-1}$ also diverges.

The Root Test

Let $\sum u_k$ be a series with positive terms and suppose

$$\rho = \lim_{k \to \infty} \sqrt[k]{u_k} = \lim_{k \to \infty} (u_k)^{1/k}$$

a) If $\rho < 1$, the series converges.

b) If $\rho > 1$ or $\rho = \infty$, the series diverges.

c) If $\rho = 1$, the series may converge or diverge.

Example 4

Use the Root Test to determine whether

$$\sum_{k=1}^{\infty}\left(\frac{4k-5}{2k+1}\right)^k$$

converges or diverges.

<u>Solution</u>:

$$u_k=\left(\frac{4k-5}{2k+1}\right)^k \text{ and } \sqrt[k]{u_k}=\sqrt[k]{\left(\frac{4k-5}{2k+1}\right)^k}=\frac{4k-5}{2k+1}$$

Example 4 (continued)

$$\rho = \lim_{k\to\infty} \sqrt[k]{u_k}$$

$$= \lim_{k\to\infty} \frac{4k-5}{2k+1} = \frac{4}{2} = 2 > 1$$

So by the Root Test, the series diverges.

Example 5

Use the Root Test to determine whether

$$\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$$

converges or diverges.

<u>Solution</u>:

$$u_k = \frac{1}{(\ln(k+1))^k} \text{ and } \sqrt[k]{u_k} = \sqrt[k]{\frac{1}{(\ln(k+1))^k}} = \frac{1}{\ln(k+1)}$$

Example 5 (continued)

$$\rho = \lim_{k\to\infty} \sqrt[k]{u_k}$$
$$= \lim_{k\to\infty} \frac{1}{\ln(k+1)} = 0 < 1$$

So by the Root Test, the series converges.