Trigonometric Double-Angle Formulas

Double Angle Identities

Two special cases of the sum of angles identities arise often enough that we choose to state these identities separately.

Identities

The double angle identities

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$\begin{aligned}\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha)\\ &= 1 - 2\sin^2(\alpha)\\ &= 2\cos^2(\alpha) - 1\end{aligned}$$

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Example 1

If $\sin(\theta) = \frac{3}{5}$ and θ is in the second quadrant, find exact values for $\sin(2\theta)$ and $\cos(2\theta)$.

To evaluate $\cos(2\theta)$, since we know the value for $\sin(\theta)$, we can use the version of the double angle that only involves sine.

$$\cos(2\theta) = 1 - 2\sin^2(\theta) = 1 - 2\left(\frac{3}{5}\right)^2 = 1 - \frac{18}{25} = \frac{7}{25}$$

Since the double angle for sine involves both sine and cosine, we'll need to first find $\cos(\theta)$, which we can do using the Pythagorean Identity.

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\left(\frac{3}{5}\right)^2 + \cos^2(\theta) = 1$$

$$\cos^2(\theta) = 1 - \frac{9}{25}$$

$$\cos(\theta) = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$$

Since θ is in the second quadrant, we know that $\cos(\theta) < 0$, so

$$\cos(\theta) = -\frac{4}{5}$$

Now we can evaluate the sine double angle

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$$

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Example 2

Simplify the expressions

a) $2\cos^2(12°)-1$ b) $8\sin(3x)\cos(3x)$

a) Notice that the expression is in the same form as one version of the double angle identity for cosine: $\cos(2\theta)=2\cos^2(\theta)-1$. Using this,

$2\cos^2(12°)-1=\cos(2\cdot 12°)=\cos(24°)$

b) This expression looks similar to the result of the double angle identity for sine.

$8\sin(3x)\cos(3x)$ Factoring a 4 out of the original expression

$4\cdot 2\sin(3x)\cos(3x)$ Applying the double angle identity

$4\sin(6x)$

Simplify $\dfrac{\cos(2t)}{\cos(t)-\sin(t)}$.

With three choices for how to rewrite the double angle, we need to consider which will be the most useful. To simplify this expression, it would be great if the denominator would cancel with something in the numerator, which would require a factor of $\cos(t)-\sin(t)$ in the numerator, which is most likely to occur if we rewrite the numerator with a mix of sine and cosine.

$\dfrac{\cos(2t)}{\cos(t)-\sin(t)}$ Apply the double angle identity

$=\dfrac{\cos^2(t)-\sin^2(t)}{\cos(t)-\sin(t)}$ Factor the numerator

$=\dfrac{(\cos(t)-\sin(t))(\cos(t)+\sin(t))}{\cos(t)-\sin(t)}$ Cancelling the common factor

$=\cos(t)+\sin(t)$ Resulting in the most simplified form

Example 3

Prove $\sec(2\alpha) = \dfrac{\sec^2(\alpha)}{2-\sec^2(\alpha)}$.

Since the right side seems a bit more complicated than the left side, we begin there.

$\dfrac{\sec^2(\alpha)}{2-\sec^2(\alpha)}$ — Rewrite the secants in terms of cosine

$$= \frac{\dfrac{1}{\cos^2(\alpha)}}{2-\dfrac{1}{\cos^2(\alpha)}}$$

At this point, we could rewrite the bottom with common denominators, subtract the terms, invert and multiply, then simplify. Alternatively, we can multiple both the top and bottom by $\cos^2(\alpha)$, the common denominator:

$$= \frac{\dfrac{1}{\cos^2(\alpha)}\cdot\cos^2(\alpha)}{\left(2-\dfrac{1}{\cos^2(\alpha)}\right)\cdot\cos^2(\alpha)}$$ Distribute on the bottom

$$= \frac{\dfrac{\cos^2(\alpha)}{\cos^2(\alpha)}}{2\cos^2(\alpha)-\dfrac{\cos^2(\alpha)}{\cos^2(\alpha)}}.$$ Simplify

$$= \frac{1}{2\cos^2(\alpha)-1}$$ Rewrite the denominator as a double angle

$$= \frac{1}{\cos(2\alpha)}$$ Rewrite as a secant

$$= \sec(2\alpha)$$ Establishing the identity

Try it Now

2. Use an identity to find the exact value of $\cos^2(75°)-\sin^2(75°)$.