Parametric Coordinates and Equations
…. Set 3

Pre Calculus	**Parametric Equations**

To this point we've looked almost exclusively at functions in the form of $y = f(x)$ or $x = h(y)$ and almost all of the formulas that we've developed require that functions be in one of these two forms. The problem is that not all curves or equations that we'd like to look at fall easily into this form. Take, for example, a circle. It is easy enough to write down the equation of a circle centered at the origin with radius r.

$$x^2 + y^2 = r^2$$

However, we will never be able to write the equation of a circle down as a single equation in either of the forms above. Sure we can solve for x or y as the following two formulas show

$$y = \pm\sqrt{r^2 - x^2} \qquad\qquad x = \pm\sqrt{r^2 - y^2}$$

but there are in fact two functions in each of these. Each formula gives a portion of the circle.

$$y = \sqrt{r^2 - x^2} \quad \text{(top)} \qquad\qquad x = \sqrt{r^2 - y^2} \quad \text{(right side)}$$

$$y = -\sqrt{r^2 - x^2} \quad \text{(bottom)} \qquad\qquad x = -\sqrt{r^2 - y^2} \quad \text{(left side)}$$

Unfortunately we usually are working on the whole circle, or simply can't say that we're going to be working only on one portion of it. Even if we can narrow things down to only one of these portions the function is still often fairly unpleasant to work with. There are also a great many curves out there that we can't even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce *parametric equations*. Instead of defining y in terms of x $(y = f(x))$ or x in terms of y $(x = h(y))$ we define both x and y in terms of a third variable called a parameter as follows,

$$x = f(t) \qquad\qquad y = g(t)$$

This third variable is usually denoted by t (as we did here) but doesn't have to be of course. Sometimes we will restrict the values of t that we'll use and at other times we won't. This will often be dependent on the problem and just what we are attempting to do. Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the *parametric curve*. Sketching a parametric curve is not always an easy thing to do.

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Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, the set of ordered pairs $(f(t), g(t))$ is a plane curve C. The equations $x = f(t)$ and $y = g(t)$ are parametric equations for C, and t is the parameter.

Examples:

(a) Sketch the parametric curve for the following set of parametric equations.

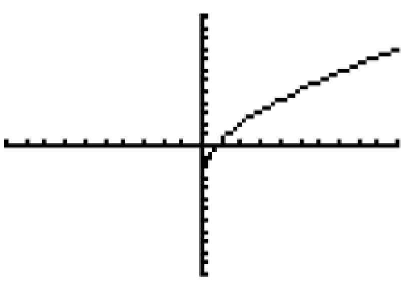
$$x = t^2 + t \qquad y = 2t - 1$$

Put your calculator in Parametric Mode: go to mode, arrow down to func (function) and then arrow over to Par, press enter. Now go to y= it should be $X_{1\ T}$ and $Y_{1\ T}$ showing. In the $X_{1\ T} = t^2 + t$ and in $Y_{1\ T} = 2t - 1$. (for T, use the X,T,θ,n button) Now 2nd graph for your table of value:

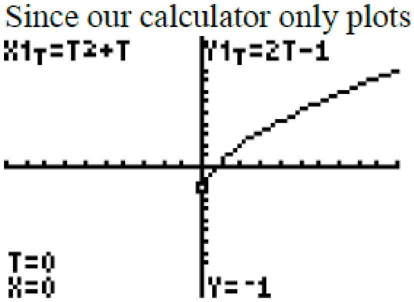
T	X_{1T}	Y_{1T}
0	0	-1
1	2	1
2	6	3
3	12	5
4	20	7
5	30	9
6	42	11

T=6

Now press graph:



Since our calculator only plots the x and y values that we see, now press the trace button:



As you see, it now gives the value for T also.

(b) Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \qquad -1 \le t \le 1$$

Now switch the window from what it is to $-1 \le t \le 1$

```
WINDOW
 Tmin=-1
 Tmax=1
 Tstep=.1308996…
 Xmin=-10
 Xmax=10
 Xscl=1
↓Ymin=-6
```

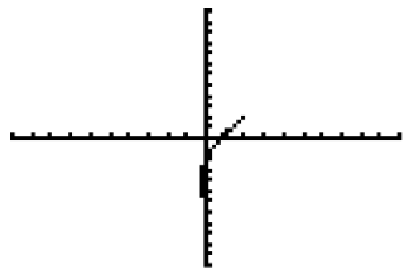
Change the tbl set to min -1 by .5:

T	X_{1T}	Y_{1T}
-1	0	-3
-.5	-.25	-2
0	0	-1
.5	.75	0
1	2	1
1.5	3.75	2
2	6	3

T=-1

The graph has changed.

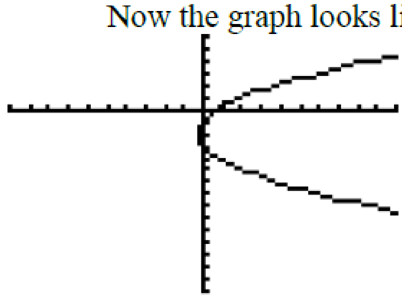
Now graph this and what do you notice



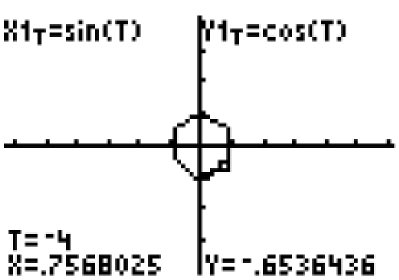
Now change it again to:

```
WINDOW
 Tmin=■4
 Tmax=3
 Tstep=.1308996…
 Xmin=-10
 Xmax=10
 Xscl=1
↓Ymin=-15
```

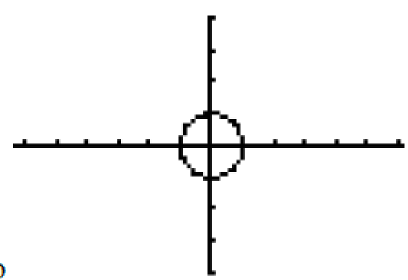
Now the graph looks like:



The calculator does not show the arrows; the path the parametric function travels but you have to place the arrows on your graph. You also should mark you values of "*t*" down at each point. You may also notice that my t-step is very small. This can change what the graph looks like also:



compared to



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Eliminating the Variable

Examples:

(a) Eliminate the parameter in the parametric equations:

$$x = t^2 - 3t \qquad y = t - 1$$

To eliminate, solve one of the equations for t and substitute that in the other equation.

$$y = t - 1$$
$$y + 1 = t$$
$$x = t^2 - 3t$$
$$x = (y+1)^2 - 3(y+1)$$

OR

Complete the square on the right side:

$$x + \frac{9}{4} = t^2 - 3t + \frac{9}{4}$$
$$x + \frac{9}{4} = \left(t - \frac{3}{2}\right)^2$$

$$\pm\sqrt{x + \frac{9}{4}} = t - \frac{3}{2}$$

$$\frac{3}{2} \pm \sqrt{x + \frac{9}{4}} = t$$

$$y = t - 1$$
$$y = \frac{3}{2} \pm \sqrt{x + \frac{9}{4}} - 1$$
$$y = \frac{1}{2} \pm \sqrt{x + \frac{9}{4}}$$

So you can solve for either x or y - sometimes one may be easier to do than the other.

(b) Eliminate the parameter in the parametric equations:

$$x = t^2 + 1 \qquad y = 2t - 1$$

Solving for t in the "x ="

$$x = t^2 + 1$$
$$x - 1 = t^2$$
$$\pm\sqrt{x-1} = t$$
$$y = 2t - 1$$
$$y = 2(\pm\sqrt{x-1}) - 1$$
$$y = \pm 2\sqrt{x-1} - 1$$

OR solving for t in the "y="

$$y = 2t - 1$$
$$y + 1 = 2t$$
$$\frac{y+1}{2} = t$$

$$x = t^2 + 1$$

$$x = \left(\frac{y+1}{2}\right)^2 + 1$$

(c) Eliminate the parameter and graph the curve represented by the parametric equations: $x = \cos t$ $\quad y = \sin t$ $\quad 0 \le t \le 2\pi$

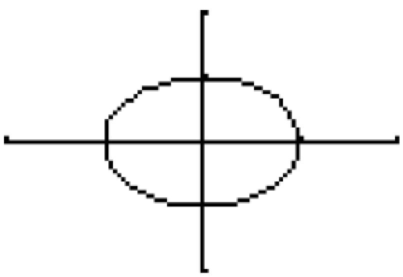
$$t = \arccos x$$
$$y = \sin(\arccos x)$$

Now depending upon what you use for you window:

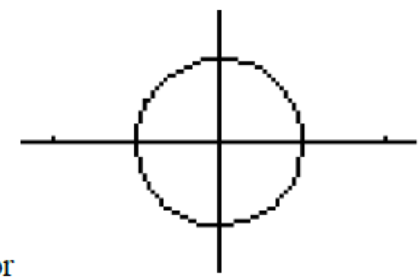
```
WINDOW
 Tmin=0
 Tmax=6.2831853…
 Tstep=.11
 Xmin=-2
 Xmax=2
 Xscl=1
↓Ymin=-2
```

or

```
WINDOW
 Tmin=0
 Tmax=6.2831853…
 Tstep=.11
 Xmin=-2.35
 Xmax=2.35
 Xscl=1
↓Ymin=-1.55
```



or



T	X_{1T}	Y_{1T}
0	1	0
90	0	1
180	-1	0
270	0	-1
360	1	0
450	0	1
540	-1	0

T=0

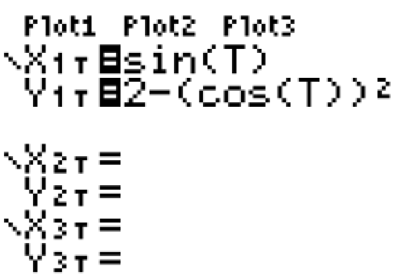
Which graph is the better representation?

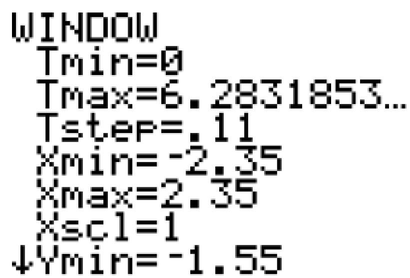
(d) Eliminate the parameter and graph the curve represented by the parametric equations: $x = \sin t$ $\qquad$ $y = 2 - \cos^2 t$

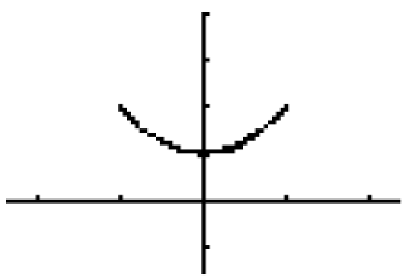
$$Recall:\ cos^2 t \ = \ 1 - \ sin^2 t$$

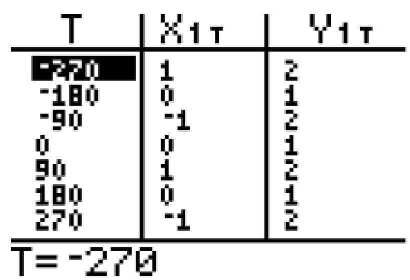
$$x = \sin t$$
$$x^2 = \ sin^2\ t$$

$$y = 2 - \ cos^2 t$$
$$y = 2 - \ (1 - sin^2 t\ )$$
$$y = 2 - 1 + \ sin^2 t$$
$$y = 1 + \ x^2$$









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Finding Parametric Equations for a Curve

Examples:

(a) Find parametric equations for the line of slope 3 that passes through the point $(2,6)$.

$$slope = M = 3; \quad x = 2 \; and \; y = 6$$
$$y = mx + b$$
$$6 = 3(2) + b$$
$$6 = 6 + b$$
$$b = 0$$

$$y = 3x$$

$$Let \quad x = t \qquad then \quad y = 3t$$

Check it. Does it work?

(b) Find a parametrization of the line through the points $A(-2,3)$ and $B(3,6)$.

Using vectors, we can solve this.

Let $point\ O\ (0,0)$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$
$$\overrightarrow{OP} = \overrightarrow{OA} + t \cdot \overrightarrow{AB}$$

$$\langle x, y \rangle = \langle -2, 3 \rangle + t\langle 3 - (-2), 6 - 3 \rangle$$

$$\langle x, y \rangle = \langle -2, 3 \rangle + t\langle 5, 3 \rangle$$

$$\langle x, y \rangle = \langle -2 + 5t, 3 + 3t \rangle$$

Therefore:

$$x = -2 + 5t$$
$$y = 3 + 3t$$