First Derivative Test … Set 4

First Derivative Tests

GOAL: To use information given by $f'(x)$ to find where $f(x)$ is increasing and decreasing, and to locate maxima and minima.

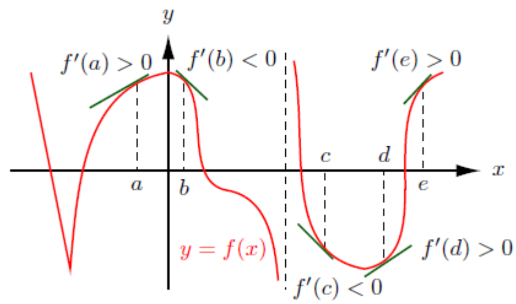


Figure 1

▶ The derivative test for increasing and decreasing functions

Q1: What does f' tell us about f?

A1: • If $f'(x) > 0$ for $\alpha < x < \beta$, then $f(x)$ is *increasing* for $\alpha < x < \beta$.

• If $f'(x) < 0$ for $\alpha < x < \beta$, then $f(x)$ is *decreasing* for $\alpha < x < \beta$.

▶ Determining the sign of $f'(x)$

What we have seen thus far: To find where f is increasing or decreasing, we need to find where $f'(x)$ is positive and negative. To do this, we start by finding the **critical points** of $f(x)$.

Definition: Critical points of the function f are points c in the domain of f where (a) $f'(c) = 0$ or (b) $f'(c)$ does not exist.

Q2: Where are the critical points of the function $f(x)$ in Figure 1? Label them $c_1, c_2, \cdots$

Remark: The only possible places (of x) where $f'(x)$ changes signs are at (i) **critical points** or (ii) where the graph has a **vertical asymptote** or is undefined.

Example 1 Find all values of x for which $f(x) = x^3 + 3x^2 - 9x + 3$ is increasing or decreasing with the steps outlined below.

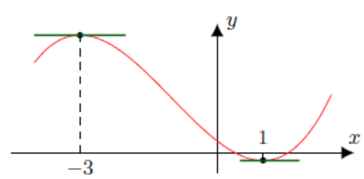
Step 1: Find all **critical points** of f. (That is all points c in the domain where $f'(c) = 0$ or $f'(c)$ does not exist.)

Since,
$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x + 3)(x - 1) = 0.$
We see that, we have the critical points at: $x = -3$ *and* $x = 1$

Step 2: Find points where f has a **vertical asymptote** or is undefined. Answer: *None*

Step 3: Draw a number line, mark all points found in Steps 1 and 2, and find the sign of $f'(x)$ in each interval between marked points.

x	$-\infty$		-3		1		$+\infty$
sign of $f'(x)$		$+$	0	$-$	0	$+$	
info about $f(x)$		↗		↘		↗	



Step 4: Write down the values of x for which f is increasing ($f'(x) > 0$) and those for which f is decreasing ($f'(x) < 0$).

From the sign table above, we have:

• f *is increasing for* $-\infty < x < -3$ *and* $1 < x < \infty$. • f *is decreasing for* $-3 < x < 1$.

First Derivative Test … Set 4

▶ First derivative test for maxima and minima

Definitions: Let c be a point in the domain of a function $f(x)$.

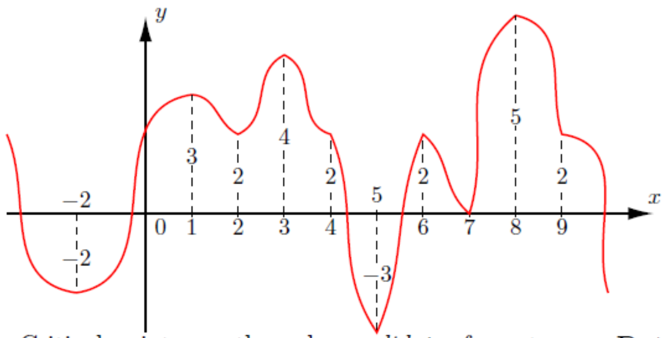
- $f(x)$ has a **local minimum** (or relative minimum) at c if $f(c) \leq f(x)$ for all x in an interval around c.
- $f(x)$ has a **local maximum** (or relative maximum) at c if $f(c) \geq f(x)$ for all x in an interval around c.

} **local extrema**

- $f(x)$ has a **global minimum** (or absolute minimum) at c if $f(c) \leq f(x)$ for all x in the domain of f.
- $f(x)$ has a **global maximum** (or absolute maximum) at c if $f(c) \geq f(x)$ for all x in the domain of f.

} **global extrema**

Example 2 Consider the following graph and locate all local and global extrema.



local minima at: $x = -2, 2, 5, 7$

local maxima at: $x = 1, 3, 6, 8$

global minimum at: $x = 5$

global maximum at: $x = 8$

Fact: Critical points are the only *candidates* for extrema. **But** you may have a critical point that's not an extremum.

> **The first derivative test for maxima and minima**
> If $f(x)$ has a critical point at c, then
> - there is a local maximum at $x = c$ if $f'(x)$ changes its sign from positive to negative, and
> - there is a local minimum at $x = c$ if $f'(x)$ changes its sign from negative to positive.

Example 3 Find all critical points of the given function and use the derivative to determine where the function is increasing, where it is decreasing, and where it has a local maximum and minimum, if any.

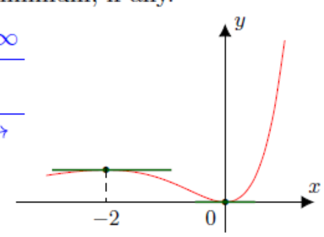
(a) $f(x) = x^2 e^x$

$f'(x) = x^2 e^x + 2xe^x$
$= xe^x(x + 2) = 0$

Critical points: $x = 0, x = -2$

x	$-\infty$		-2		0		$+\infty$
sign of $f'(x)$		$+$	0	$-$	0	$+$	
info about $f(x)$		↗		↘		↗	

local max at $x = -2$ *local min at* $x = 0$



(b) $f(x)$ is such that the graph of the **derivative** of $f(x)$ is given below.

- *Critical points:* $x = -1, 1, 2, 3$.
- *Increasing:* $(-1, 1), (2, 3), (3, \infty)$.
- *Decreasing:* $(-\infty, -1), (1, 2)$.
- *Local maximum at* $x = 1$.
- *Local minimum at* $x = -1, 2$.