Differentials

For a while now, we have been using the notation

$$\frac{dy}{dx}$$

to mean the derivative of y with respect to x. Here x is any variable, and y is a variable whose value depends on x.

One of the reasons that we like this notation is that it suggests the meaning of the derivative. The quantities dx and dy are called **differentials**, and represent very small changes in the values of x and y. Specifically, if we change x by a small amount dx, then y will change by a small amount dy, and the ratio dy/dx is the derivative.

It has been a while since we discussed these ideas. The following example should help you to remember:

EXAMPLE 1 At a certain instant, the value of x is 3, and the value of y is 5. A short time later, the value of x has increased to 3.01, and the value of y has increased to 5.04. Estimate dy/dx.

SOLUTION The small increase in x is

$$dx = 3.01 - 3 = 0.01,$$

and the corresponding increase in y is

$$dy = 5.04 - 5 = 0.04.$$

Therefore

$$\frac{dy}{dx} \approx \frac{0.04}{0.01} = 4.$$

However, the calculation above does make sense as an approximation. Even though 0.01 isn't really infinitesimal, it is very small, so treating it as infinitesimal ought to yield answers that are approximately right. If we want the approximation to be more accurate, we would need to use a smaller change in x, such as 0.001 or 0.0001.

We discussed all of these ideas once before. The idea of better and better approximations leads naturally to the idea of as limit. Indeed, it is possible to define the derivative entirely using limits:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This modern "limit definition" of the derivative eschews differentials and infinitesimals, relying instead on the more concrete notion of a limit. Before the limit definition of the derivative, mathematicians spent a century and a half arguing about the legitimacy of infinite and infinitesimal numbers, and the legitimacy of calculus itself. The limit definition puts those objections to rest, and provides a solid foundation for modern calculus.

However, the idea of infinitesimal numbers remains. Though calculus is now based on limits, many problems are most easily solved using infinitesimals, and reasoning using infinitesimals can be a very powerful technique. Ultimately, such reasoning needs to be justified using limits, but in practice infinitesimals are often easier to understand and easier to use.

Equations Involving Differentials

A **differential** is a variable whose value is infinitesimal. By convention, all differentials are preceded by the letter "*d*", which usually means something like "little bit of" or "little change in".

Any equation that involves derivatives can also be written as an equation involving differentials. For example, if a square has side length x, then the area of the square is given by the formula

$$A = x^2.$$

Taking the derivative with respect to x yields the equation

$$\frac{dA}{dx} = 2x.$$

This equation involves a derivative, which is really a ratio of two differentials. If we multiply through by dx, we get an equation relating the two differentials:

$$dA = 2x\,dx.$$

Note that both sides of this equation are infinitesimal. This equation tells us how much the area of the square will change if we increase the side length by a small amount. For example, if x is 4 and we increase x by 0.003, then the change in A will be approximately

$$2(4)(0.003) = 0.024.$$

Differentials

Here is a summary of this technique:

Formula for dy in terms of x and dx.

Let x and y be variables, where $y = f(x)$. To find a formula for dy in terms of x and dx, start by taking the derivative with respect to x:

$$\frac{dy}{dx} = f'(x).$$

Next, multiply through by dx to obtain the desired formula:

$$dy = f'(x)\,dx.$$

Differentials

EXAMPLE 2 Suppose that $y = x^3 + 4x$.

(a) Find a formula for dy in terms of x and dx.

(b) Suppose we increase x from 3 to 3.02. Use differentials to estimate the corresponding increase in the value of y.

SOLUTION Taking the derivative of the given formula yields

$$\frac{dy}{dx} = 3x^2 + 4.$$

We can now multiply through by dx to get

$$dy = (3x^2 + 4)\,dx.$$

This answers part (a). For part (b), we substitute $x = 3$ and $dx = 0.02$ to get

$$dy = \big(3(3)^2 + 4\big)(0.02) = 0.62.$$

Differentials

In science, differentials are often used to estimate the possible error in the value of a variable obtained through calculation. The following example illustrates this technique:

EXAMPLE 3 The energy stored in a certain capacitor obeys the formula

$$E = \frac{1}{2}CV^2$$

where V is the voltage difference across the leads, and $C = 0.15$ Joules/volt^2.

(a) Find a formula for dE in terms of V and dV.

(b) An engineer measures the voltage across the leads as 2.8 volts, with an error of ± 0.05 volts. Find the energy stored in the capacitor, and estimate the error in your answer.

SOLUTION Taking the derivative with respect to V yields

$$\frac{dE}{dV} = CV.$$

We can now multiply through by dV to get

$$dE = CV\,dV.$$

This answers part (a). For part (b), the energy stored in the capacitor is

$$E = \frac{1}{2}CV^2 = \frac{1}{2}(0.15)(2.8)^2 = 0.588 \text{ Joules.}$$

To estimate the error, we imagine what happens if we change V by $dV = \pm 0.05$ volts. Using our differentials formula,

$$dE = CV\,dV = (0.15)(2.8)(\pm 0.05) = \pm 0.021 \text{ Joules,}$$

This is roughly the error in the value of the energy.