3.1 Continuity and the Intermediate Value Theorem

If a continuous function on a closed interval has opposite signs at the endpoints, it must be zero at some interior point.

In Section 1.2, we defined continuity as follows: "A function $f(x)$ is said to be continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$." A function is said to be *continuous on a given interval* if it is continuous at every point on that interval. If a function f is continuous on the whole real line, we just say that "f is continuous." An imprecise but useful guide is that a function is continuous when its graph can be drawn "without removing pencil from paper." In Figure 3.1.1, the curve on the left is continuous at x_0 while that on the right is not.

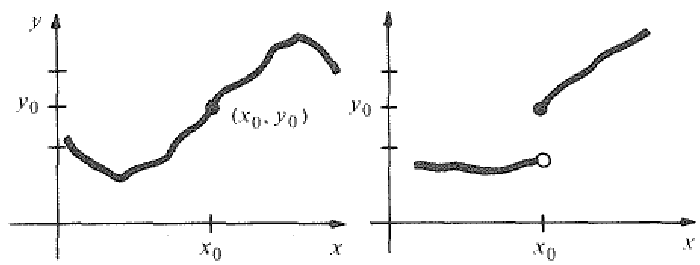

Figure 3.1.1. Illustrating a continuous curve (left) and a discontinuous curve (right).

Example 1 Decide where each of the functions whose graphs appear in Fig. 3.1.2 is continuous. Explain your answers.

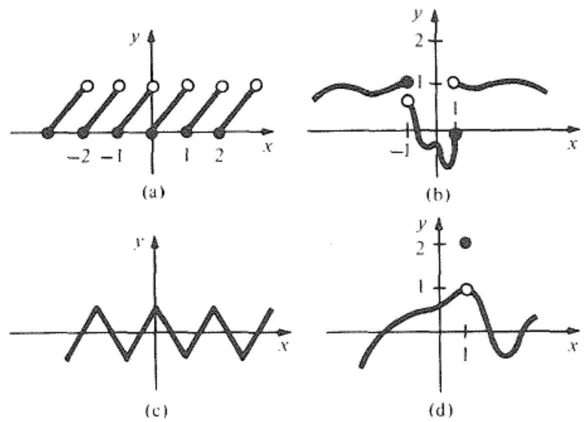

Figure 3.1.2. Where are these functions continuous?

Solution

(a) This function jumps in value at each of the points $x_0 = 0$, $x_0 = \pm 1$, $x_0 = \pm 2, \ldots$, so $\lim_{x \to x_0} f(x)$ does not exist at these points and thus f is not continuous there; however, f is continuous on each of the intervals between the jump points.

(b) This function jumps in value at $x_0 = -1$ and $x_0 = +1$, and so $\lim_{x \to +1} f(x)$ and $\lim_{x \to -1} f(x)$ do not exist. Thus f is not continuous at $x_0 = \pm 1$; it is continuous on each of the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

(c) Even though this function has sharp corners on its graph, it *is* continuous; $\lim_{x \to x_0} f(x) = f(x_0)$ at each point x_0.

(d) Here $\lim_{x \to 1} f(x) = 1$, so the limit exists. However, the limit does not equal $f(1) = 2$. Thus f is *not* continuous at $x_0 = 1$. It is continuous on the intervals $(-\infty, 1)$ *and* $(1, \infty)$. ▲

In Section 1.2, we used various limit theorems to establish the continuity of functions that are basic to calculus. For example, the rational function rule for limits says that a *rational function is continuous at points where its denominator does not vanish.*

Example 2 Show that the function $f(x) = (x - 1)/3x^2$ is continuous at $x_0 = 4$.

Solution This is a rational function whose denominator does not vanish at $x_0 = 4$, so it is continuous by the rational function rule. ▲

Example 3 Let $g(x)$ be the *step function* defined by

$$g(x) = \begin{cases} 0 & \text{if} \quad x \leqslant 0, \\ 1 & \text{if} \quad x > 0. \end{cases}$$

Show that g is not continuous at $x_0 = 0$. Sketch.

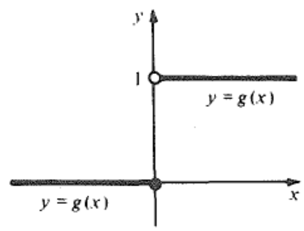

Figure 3.1.3. This step function is discontinuous at $x_0 = 0$.

Solution The graph of g is shown in Fig. 3.1.3. Since g approaches (in fact, equals) 0 as x approaches 0 from the left, but approaches 1 as x approaches 0 from the right, $\lim_{x \to 0} g(x)$ does not exist. Therefore, g is not continuous at $x_0 = 0$. ▲

Example 4 Using the laws of limits, show that if f and g are continuous at x_0, so is fg.

Solution We must show that $\lim_{x\to x_0}(fg)(x) = (fg)(x_0)$. By the product rule for limits, $\lim_{x\to x_0}[f(x)g(x)] = [\lim_{x\to x_0} f(x)][\lim_{x\to x_0} g(x)] = f(x_0)g(x_0)$, since f and g are continuous at x_0. But $f(x_0)g(x_0) = (fg)(x_0)$, and so $\lim_{x\to x_0}(fg)(x) = (fg)(x_0)$, as required. ▲

In Section 1.3, we proved the following theorem: *if f is differentiable at x_0, then f is continuous at x_0*. Using our knowledge of differential calculus, we can use this relationship to establish the continuity of additional functions or to confirm the continuity of functions originally determined using the laws of limits.

Example 5 (a) Show that $f(x) = 3x^2/(x^3 - 2)$ is continuous at $x_0 = 1$. Where else is it continuous?

(b) Show that $f(x) = \sqrt{x^2 + 2x + 1}$ is continuous at $x = 0$.

Solution (a) By our rules for differentiation, we see that this function is differentiable at $x_0 = 1$; indeed, $x^3 - 2$ does not vanish at $x_0 = 1$. Thus f is also continuous at $x_0 = 1$. Similarly, f is continuous at each x_0 such that $x_0^3 - 2 \neq 0$, i.e., at each $x_0 \neq \sqrt[3]{2}$.

(b) This function is the composition of the square root function $h(u) = \sqrt{u}$ and the function $g(x) = x^2 + 2x + 1$; $f(x) = h(g(x))$. Note that $g(0) = 1 > 0$. Since g is differentiable at any x (being a polynomial), and h is differentiable at $u = 1$, f is differentiable at $x = 0$ by the chain rule. Thus f is continuous at $x = 0$. ▲

According to our previous discussion, a continuous function is one whose graph never "jumps." The definition of continuity is *local* since continuity at each point involves values of the function only near that point. There is a corresponding *global* statement, called the intermediate value theorem, which involves the behavior of a function over an entire interval $[a, b]$.

Intermediate Value Theorem (First Version)

Let f be continuous[1] on $[a, b]$ and suppose that, for some number c, $f(a) < c < f(b)$ or $f(a) > c > f(b)$. Then there is some point x_0 in (a, b) such that $f(x_0) = c$.

[1] Our definition of continuity on $[a, b]$ assumes that f is defined near each point $\bar{x}$ of $[a, b]$, including the endpoints, and that $\lim_{x \to \bar{x}} f(x) = f(\bar{x})$. Actually, at the endpoints, it is enough to assume that the one-sided limits (from inside the interval) exist, rather than the two-sided ones.

Exercises for Section 3.1

1. Decide where each of the functions whose graph is sketched in Fig. 3.1.6 is continuous.

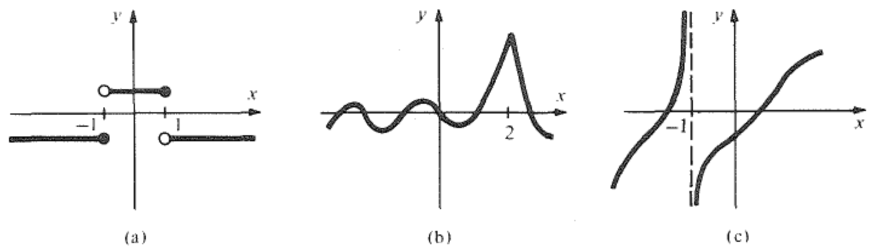

Figure 3.1.6. Where are these continuous?

2. Which of the functions in Fig. 3.1.7 are continuous at $x_0 = 1$?

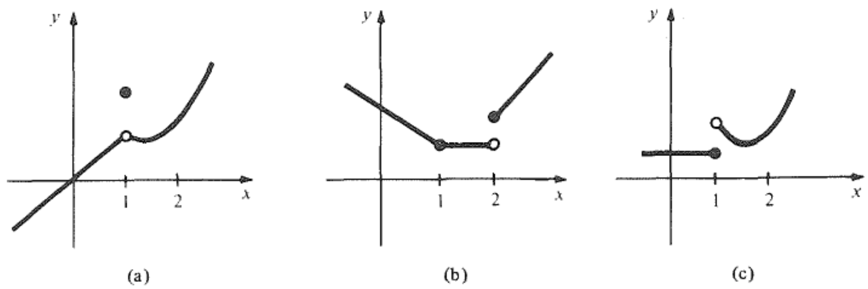

Figure 3.1.7. Which of these functions are continuous at $x_0 = 1$?

3. Show that $f(x) = (x^2 + 1)(x^2 - 1)$ is continuous at $x_0 = 0$.
4. Show that $f(x) = x^3 + 3x^2 - 2x$ is continuous at $x_0 = 3$.
5. Prove that $(x^2 - 1)/(x^3 + 3x)$ is continuous at $x_0 = 1$.
6. Prove that $(x^4 - 8)/(x^3 + 2)$ is continuous at $x_0 = 0$.
7. Where is $(x^2 - 1)/(x^4 + x^2 + 1)$ continuous?
8. Where is $(x^4 + 1)/(x^3 - 8)$ continuous?
9. Let $f(x) = (x^3 + 2)/(x^2 - 1)$. Show that f is continuous on $[-\frac{1}{2}, \frac{1}{2}]$.
10. Is the function $(x^3 - 1)/(x^2 - 1)$ continuous at 1? Explain your answer.
11. Let $f(x)$ be the step function defined by

$$f(x) = \begin{cases} -1 & \text{if} \quad x < 0, \\ -2 & \text{if} \quad x \geqslant 0. \end{cases}$$

Show that f is discontinuous at 0.
12. Let $f(x)$ be the *absolute value function*: $f(x) = |x|$; that is,

$$f(x) = \begin{cases} x & \text{if} \quad x \geqslant 0, \\ -x & \text{if} \quad x < 0. \end{cases}$$

Show that f is continuous at $x_0 = 0$.
13. Using the laws of limits, show that if f and g are continuous at x_0, so is $f + g$.
14. If f and g are continuous at x_0 and $g(x_0) \neq 0$, show that f/g is continuous at x_0.
15. Where is $f(x) = 8x^3/\sqrt{x^2 - 8}$ continuous?
16. Where is $f(x) = 9x^2 - 3x/\sqrt{x^4 - 2x^2 - 8}$ continuous?
17. Show that the equation $-s^5 + s^2 = 2s - 6$ has a solution.
18. Prove that the equation $x^3 + 2x - 1 = 7$ has a solution.
19. Prove that $f(x) = x^8 + 3x^4 - 1$ has at least two distinct zeros.
20. Show that $x^4 - 5x^2 + 1$ has at least two distinct zeros.
21. The roots of $f(x) = x^3 - 2x - x^2 + 2$ are $\sqrt{2}$, $-\sqrt{2}$, and 1. By evaluating $f(-3)$, $f(0)$, $f(1.3)$, and $f(2)$, determine the sign of $f(x)$ on each of the intervals between its roots.
22. Use the method of bisection to approximate $\sqrt{7}$ to within two decimal places. [*Hint:* Let $f(x) = x^2 - 7$. What should you use for a and b?]
23. Find a solution of the equation $x^5 - x = 3$ to an accuracy of 0.01.
24. Find a solution of the equation $x^5 - x = 5$ to an accuracy of 0.1.
25. Suppose that f is continuous on $[-1, 1]$ and that $f(x) - 2$ is never zero on $[-1, 1]$. If $f(0) = 0$, show that $f(x) < 2$ for all x in $[-1, 1]$.
26. Suppose that f is continuous on $[3, 5]$ and that $f(x) \neq 4$ for all x in $[3, 5]$. If $f(3) = 3$, show that $f(5) < 4$.
27. Let $f(x)$ be 1 if a certain sample of lead is in the solid state at temperature x; let $f(x)$ be 0 if it is in the liquid state. Define x_0 to be the melting point of this lead sample. Is there any way to define $f(x_0)$ so as to make f continuous? Give reasons for your answer, and supply a graph for f.
28. An empty bucket with a capacity of 10 liters is placed beneath a faucet. At time $t = 0$, the faucet is turned on, and water flows from the faucet at the rate of 5 liters per minute. Let $V(t)$ be the volume of the water in the bucket at time t. Present a plausible argument showing that V is a continuous function on $(-\infty, \infty)$. Sketch a graph of V. Is V differentiable on $(-\infty, \infty)$?
29. Let $f(x) = (x^2 - 4)/(x - 2)$, $x \neq 2$. Define $f(2)$ so that the resulting function is continuous at $x = 2$.
30. Let $f(x) = (x^3 - 1)/(x - 1)$, $x \neq 1$. How should $f(1)$ be defined in order that f be continuous at each point?
31. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 + 1 & \text{if} \quad x < 1, \\ ? & \text{if} \quad 1 \leqslant x \leqslant 3, \\ x - 6 & \text{if} \quad 3 < x. \end{cases}$$

How can you define $f(x)$ on the interval $[1, 3]$ in order to make f continuous on $(-\infty, \infty)$? (A geometric argument will suffice.)
32. Let $f(x) = x + (4/x)$ for $x \leqslant -\frac{1}{2}$ and $x \geqslant 2$. Define $f(x)$ for x in $(-\frac{1}{2}, 2)$ in such a way that the resulting function is continuous on the whole real line.
33. Let $f(x)$ be defined by $f(x) = (x^2 - 1)/(x - 1)$ for $x \neq 1$. How should you define $f(1)$ to make the resulting function continuous? [*Hint:* Plot a graph of $f(x)$ for x near 1 by factoring the numerator.]
34. Let $f(x)$ be defined by $f(x) = 1/x$ for $x \neq 0$. Is there any way to define $f(0)$ so that the resulting function will be continuous?
35. Sketch the graph of a function which is continuous on the whole real line and differentiable everywhere except at $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.
36. Is a function which is continuous at x_0 necessarily differentiable there? Prove or give a counterexample.
37. The function $f(x) = 1/(x - 1)$ never takes the value zero, yet $f(0) = -1$ is negative and $f(2) = 1$ is positive. Why isn't this a counterexample to the intermediate value theorem?
38. "Prove" that you were once exactly one meter tall.

3.2 Increasing and Decreasing Functions

The sign of the derivative indicates whether a function is increasing or decreasing.

We begin this section by defining what it means for a function to be increasing or decreasing. Then we show that a function is increasing when its derivative is positive and is decreasing when its derivative is negative. Local maximum and minimum points occur where the derivative changes sign.

We can tell whether a function is increasing or decreasing at x_0 by seeing how its graph crosses the horizontal line $y = f(x_0)$ at x_0 (see Fig. 3.2.1). This

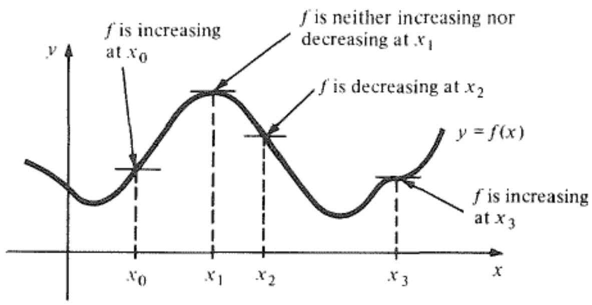

Figure 3.2.1. Places where the function f is increasing and decreasing.

geometric picture is the basis of the precise definition of increasing and decreasing. Note that at the point x_0 in Fig. 3.2.1, $f(x)$ is less than $f(x_0)$ for x just to the left of x_0, while $f(x)$ is larger than $f(x_0)$ for x just to the right. We cannot take x too far to the right, as the figure shows. The following paragraph gives the technical definition.

We say that a function f is *increasing* at x_0 if there is an interval (a, b) containing x_0 such that:

1. If $a < x < x_0$, then $f(x) < f(x_0)$.
2. If $x_0 < x < b$, then $f(x) > f(x_0)$.

Similarly, f is *decreasing* at x_0 if there is an interval (a, b) containing x_0 such that:

1. If $a < x < x_0$, then $f(x) > f(x_0)$.
2. If $x_0 < x < b$, then $f(x) < f(x_0)$.

The purpose of the interval (a, b) is to limit our attention to a small region about x_0. Indeed, the notions of increasing and decreasing at x_0 are local; they depend only on the behavior of the function near x_0. In examples done "by hand," such as Example 1 below, we must actually find the interval (a, b). We will soon see that calculus provides an easier method of determining where a function is increasing and decreasing.