Augmented Matrices … Set 1

THEOREM 1 Operations That Produce Row-Equivalent Matrices

An augmented matrix is transformed into a row-equivalent matrix by performing any of the following **row operations**:

(A) Two rows are interchanged $(R_i \leftrightarrow R_j)$.

(B) A row is multiplied by a nonzero constant $(kR_i \rightarrow R_i)$.

(C) A constant multiple of one row is added to another row $(kR_j + R_i \rightarrow R_i)$.

Note: The arrow $\rightarrow$ means "replaces."

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EXAMPLE 1 Solving a System Using Augmented Matrix Methods Solve using augmented matrix methods:

$$\begin{aligned} 3x_1 + 4x_2 &= 1 \\ x_1 - 2x_2 &= 7 \end{aligned} \qquad (4)$$

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Answers

SOLUTION We start by writing the augmented matrix corresponding to system (4):

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \tag{5}$$

Our objective is to use row operations from Theorem 1 to try to transform matrix (5) into the form

$$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right] \tag{6}$$

where m and n are real numbers. Then the solution to system (4) will be obvious, since matrix (6) will be the augmented matrix of the following system (a row in an augmented matrix always corresponds to an equation in a linear system):

$$\begin{aligned} x_1 &= m \qquad & x_1 + 0x_2 &= m \\ x_2 &= n \qquad & 0x_1 + x_2 &= n \end{aligned}$$

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Answers

Now we use row operations to transform matrix (5) into form (6).

Step 1 To get a 1 in the upper left corner, we interchange R_1 and R_2 (Theorem 1A):

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$$

Step 2 To get a 0 in the lower left corner, we multiply R_1 by (-3) and add to R_2 (Theorem 1C)—this changes R_2 but not R_1. Some people find it useful to write $(-3R_1)$ outside the matrix to help reduce errors in arithmetic, as shown:

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right] \underset{\sim}{(-3)R_1 + R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

$-3 \quad 6 \quad -21$

Step 3 To get a 1 in the second row, second column, we multiply R_2 by $\frac{1}{10}$ (Theorem 1B):

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right] \underset{\sim}{\tfrac{1}{10}R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

Step 4 To get a 0 in the first row, second column, we multiply R_2 by 2 and add the result to R_1 (Theorem 1C)—this changes R_1 but not R_2:

$0 \quad 2 \quad -4$

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right] \underset{\sim}{2R_2 + R_1 \rightarrow R_1} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

We have accomplished our objective! The last matrix is the augmented matrix for the system

$$\begin{aligned} x_1 &= 3 \qquad & x_1 + 0x_2 &= 3 \\ x_2 &= -2 \qquad & 0x_1 + x_2 &= -2 \end{aligned} \qquad (7)$$

Since system (7) is equivalent to system (4), our starting system, we have solved system (4); that is, $x_1 = 3$ and $x_2 = -2$.

CHECK

$$\begin{aligned} 3x_1 + 4x_2 &= 1 \qquad & x_1 - 2x_2 &= 7 \\ 3(3) + 4(-2) &\stackrel{?}{=} 1 \qquad & 3 - 2(-2) &\stackrel{?}{=} 7 \\ 1 &\stackrel{\checkmark}{=} 1 \qquad & 7 &\stackrel{\checkmark}{=} 7 \end{aligned}$$

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Answers

The preceding process may be written more compactly as follows:

Step 1: Need a 1 here.

$$\begin{bmatrix}3 & 4 & | & 1\\ 1 & -2 & | & 7\end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

Step 2: Need a 0 here.

$$\sim\begin{bmatrix}1 & -2 & | & 7\\ 3 & 4 & | & 1\end{bmatrix} \qquad (-3)R_1 + R_3 \to R_2$$

$$-3 \quad 6 \quad -21$$

Step 3: Need 1 here.

$$\sim\begin{bmatrix}1 & -2 & | & 7\\ 0 & 10 & | & -20\end{bmatrix} \qquad \tfrac{1}{10}R_2 \to R_2$$

Step 4: Need a 0 here.

$$0 \quad 2 \quad -4$$

$$\sim\begin{bmatrix}1 & -2 & | & 7\\ 0 & 1 & | & -2\end{bmatrix} \qquad 2R_2 + R_1 \to R_1$$

$$\sim\begin{bmatrix}1 & 0 & | & 3\\ 0 & 1 & | & -2\end{bmatrix}$$

Therefore, $x_1 = 3$ and $x_2 = -2$.

Matched Problem 1 Solve using augmented matrix methods:

$$\begin{aligned} 2x_1 - x_2 &= -7\\ x_1 + 2x_2 &= 4 \end{aligned}$$